Analysis of Error Floors of Non-binary LDPC Codes over MBIOS channel

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Outline

1. Background
   - Non-binary low-density parity-check (LDPC) codes
   - Existing methods for lowering error floors
   - Channel model
2. Lowerling decoding error rates in the error floor over MBIOS channel
   - Condition for successful decoding
   - Code design method to lower error floor
   - Simulation results
3. Analysis of symbol error rates in error floors
   - Lower bound on symbol error rate in error floor
   - Simulation results
4. Conclusion and related works

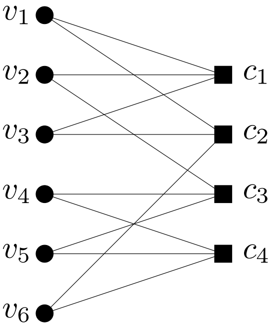
Outline

Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

Linear code defined by a sparse parity check matrix $H \in \mathbb{F}_{2^m}^{M \times N}$

$$C := \{\mathbf{x} \in \mathbb{F}_{2^m}^N \mid H\mathbf{x} = \mathbf{0}\}$$



$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\ 0 & h_{3,2} & 0 & h_{3,4} & h_{3,5} & 0 \\ 0 & 0 & 0 & h_{4,4} & h_{4,5} & h_{4,6} \end{pmatrix} \end{array}$$

Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

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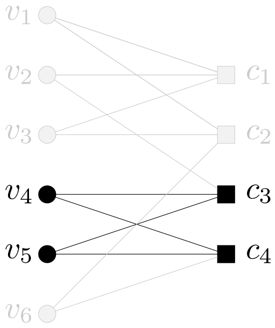

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Zigzag cycle

Cycle constructed by the variable nodes of degree 2

Zigzag cycles degrade decoding performance

- Error floors are mainly caused by codewords of small weight.

Example : Zigzag cycle yields codeword

α : primitive element of $\mathbb{F}_{2^m}$

$$H = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\ 0 & h_{3,2} & 0 & 1 & \alpha^4 & 0 \\ 0 & 0 & 0 & \alpha^6 & \alpha^{10} & h_{4,6} \end{pmatrix}$$

$$(0 \;\; 0 \;\; 0 \;\; \alpha^4 \;\; 1 \;\; 0) \in C$$

$\det \begin{pmatrix} 1 & \alpha^4 \\ \alpha^6 & \alpha^{10} \end{pmatrix} = 0$

Zigzag cycle yields codeword.

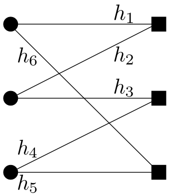
Cycle cancellation [Poulliat 2008]

Existing method : Cycle cancellation [Poulliat 2008]

Design edge labels in zigzag cycle so that

$$\det \tilde{H} \neq 0.$$

$$\tilde{H} = \begin{pmatrix} h_1 & h_2 & 0 \\ 0 & h_3 & h_4 \\ h_6 & 0 & h_5 \end{pmatrix}$$



$$\det \tilde{H} = h_1 h_3 h_5 + h_2 h_4 h_6 \neq 0 \iff \beta = h_1^{-1} h_2 h_3^{-1} h_4 h_5^{-1} h_6 \neq 1$$

$$\iff \beta = \prod_{i=1}^{3} h_{2i-1}^{-1} h_{2i} \neq 1$$

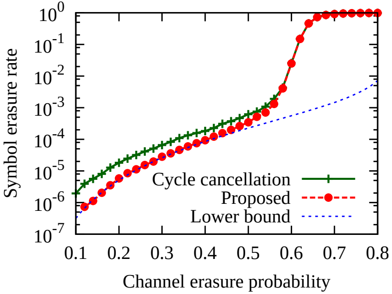
[Poulliat 2008] C. Poulliat, M. Fossorier and D. Declercq, "Design of regular (2, d_c)-LDPC codes over GF(q) using their binary images", IEEE Trans. Comm. (2008)

Binary erasure channel (BEC) case [ISIT 2010]

Existing method: Modified cycle cancellation [ISIT 2010]

Design edge labels in zigzag cycle so that

$$\beta \notin \bigcup_{r>0:r|m, r\neq m} \left\{ \alpha^{i\frac{2^m-1}{2^r-1}} \mid i = 0, 1, \ldots, 2^r - 2 \right\}$$



Symbol code length : 315
Order of Galois field : 16

[ISIT 2010] T. Nozaki, K. Kasai and K. Sakaniwa, “Error floors of non-binary LDPC codes,” ISIT 2010

Channel Model

Memoryless binary-input output-symmetric (MBIOS) channel

- Memoryless

$$p_{Y|X}(y \mid x) = \prod_t p_{Y_t|X_t}(y_t \mid x_t)$$

- symmetric

$$p_{Y_t|X_t}(y_t \mid 0) = p_{Y_t|X_t}(-y_t \mid 1)$$

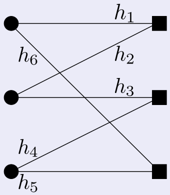
Example : Member of MBIOS channel

- Binary erasure channel (BEC)
- Binary symmetric channel (BSC)
- Additive white Gaussian noise (AWGN) channel

Outline

Condition for successful decoding

Definition : zigzag cycle code



symbol code length 3

$\beta = \prod_{i=1}^{s} h_{2i-1}^{-1} h_{2i}$

$\gamma_i = h_{2i-1}^{-1} h_{2i}$

s : symbol code length

σ : the order of β

$C_k(\gamma)$: γ-th entry of the initial message of k-th variable node

Lemma 1 : Condition for successful decoding

BP decoding succeeds for zigzag cycle code of length s

$\iff$ for all $x \in A_\beta := \{\alpha^j \mid j = 0, 1, \ldots, \frac{2^m-1}{\sigma} - 1\}$

$$\prod_{k=1}^{s} (C_k(0))^\sigma > \prod_{t=0}^{\sigma-1} \prod_{k=1}^{s} C_k\left(\beta^t x \prod_{s=1}^{k-1} \gamma_s\right).$$

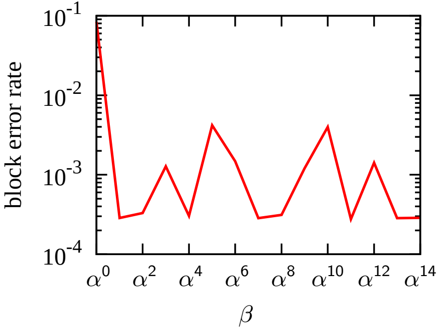
Condition for the lowest decoding error rate

Theorem 1

The zigzag cycles with cycle parameter $\beta \notin \mathcal{H}_m$ have good decoding performance, where

$$\mathcal{H}_m := \bigcup_{0<r<2^m-1 : r | 2^m-1} \left\{ \alpha^{i\frac{2^m-1}{r}} \mid i = 0, \ldots, r-1 \right\}.$$

Simulation result



additive white Gaussian noise (AWGN) channel

$\sigma^2 = 1$

Zigzag cycle code

Symbol code length : 3

Order of Galois field : 16

Construction of code

Proposition 1: Design method to lower error floor for MBIOS channel

Design edge labels in zigzag cycle so that

$$\beta \notin \mathcal{H}_m = \bigcup_{0<r<2^m-1:r|2^m-1} \left\{ \alpha^{i\frac{2^m-1}{r}} \mid i = 0, \ldots, r-1 \right\}.$$

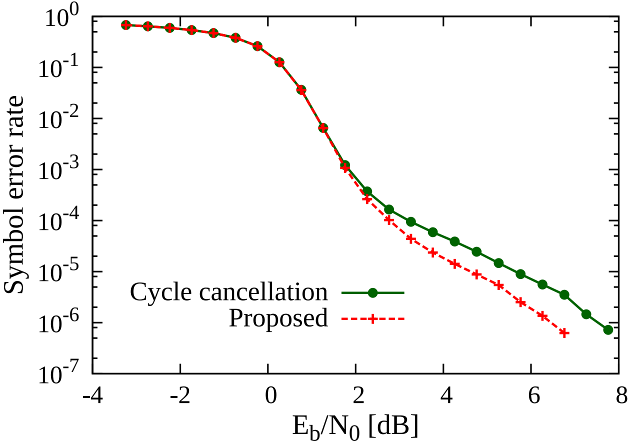
Note : BEC case

Design edge labels in zigzag cycle so that

$$\beta \notin \bigcup_{r>0:r|m, r\neq m} \left\{ \alpha^{i\frac{2^m-1}{2^r-1}} \mid i = 0, \ldots, r-1 \right\}.$$

Field	The elements of $\mathcal{H}_m$
$\mathbb{F}_{2^4}$	$1, \alpha^3, \alpha^5, \alpha^6, \alpha^9, \alpha^{10}, \alpha^{12}$
$\mathbb{F}_{2^6}$	$1, \alpha^3, \alpha^6, \alpha^7, \alpha^9, \alpha^{12}, \alpha^{14}, \alpha^{15}, \alpha^{18}, \alpha^{21}, \alpha^{24}, \alpha^{25}, \alpha^{27}, \alpha^{28}, \alpha^{30},$ $\alpha^{33}, \alpha^{35}, \alpha^{36}, \alpha^{39}, \alpha^{42}, \alpha^{45}, \alpha^{48}, \alpha^{49}, \alpha^{51}, \alpha^{54}, \alpha^{56}, \alpha^{57}, \alpha^{60}$

Simulation result (AWGN channel)

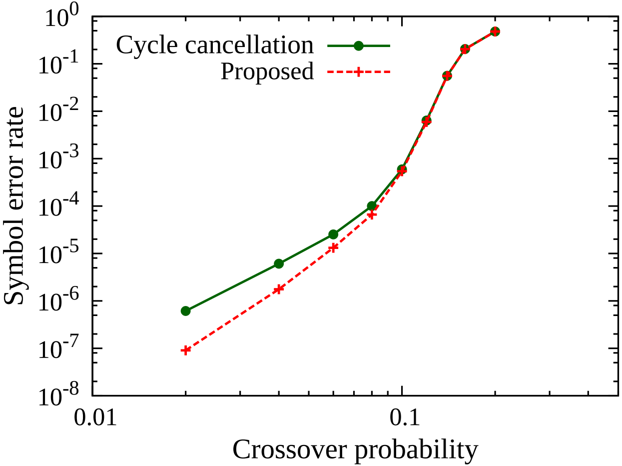


(2,3)-regular LDPC code ensemble

Symbol code length : 315 symbols

Order of Galois field : 16

Simulation result (BSC)



(2,3)-regular LDPC code ensemble
Symbol code length : 315 symbols
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Outline

Condition for successful decoding for zigzag cycle codes

Definition : Log-likelihood ratio

$Y_{v,i}$: i-th channel output in v-th variable node

$$Z_{v,i} := \log \frac{\Pr(Y_{v,i} \mid X_{v,i} = 0)}{\Pr(Y_{v,i} \mid X_{v,i} = 1)}.$$

All zero assumption

The decoding error rate does not depend on the codeword.

Lemma2 : Condition for successful decoding

For zigzag cycle codes with cycle parameter $\beta \notin \mathcal{H}_m$ under BP decoding

All the symbols are successfully decoded $\iff \sum_{v,i} Z_{v,i} > 0$.

All the errors are not recoverable $\iff \sum_{v,i} Z_{v,i} \leq 0$.

The decoding error probablities of all the zigzag cycle codes are lower bounded by $\Pr(\sum_{v,i} Z_{v,i} \geq 0)$

Analysis of error floors

Theorem 2 : Error floor for NB-LDPC codes

N : symbol code length
$\lambda(x) = \sum_i \lambda_i x^{i-1}$, $\rho(x) := \sum_i \rho_i x^{i-1}$: degree distribution pair
$\mu := \lambda'(0)\rho'(1)$
s_{g} : expurgation parameter (no cycle of weight in $\{1, 2, \ldots, s_{\mathrm{g}} - 1\}$)
$Z^{(sm)} := \sum_{v=1}^{s} \sum_{i=1}^{m} Z_{v,i}$

Let $\mathrm{P_s}(\mathcal{E})$ be the symbol error rate of the code ensemble designed by proposed method over the MBIOS channel charactrized by its L-density a under BP decoding.
For $\int \mathsf{a}(x) \exp[-x/2] \mathrm{d}x < \mu^{-\frac{1}{m}}$ and the sufficiently large N, we have

$$\mathrm{P_s}(\mathcal{E}) \geq \frac{1}{2N} \sum_{s=s_{\mathrm{g}}}^{\infty} \mu^s \mathrm{Pr}(Z^{(sm)} \leq 0) + o\left(\frac{1}{N}\right).$$

Analysis of error floors

Corollary1 : AWGN channel case

For AWGN channel with channel variance σ^2

$$\mathrm{P_s}(\mathcal{E}) \geq \frac{1}{2N} \sum_{s=s_{\mathrm{g}}}^{\infty} \mu^s Q\left(\frac{\sqrt{sm}}{\sigma}\right) + o\left(\frac{1}{N}\right), \quad \text{for} \quad \sigma^2 < \begin{cases} \infty & \text{for} \quad \mu \leq 1 \\ \frac{m}{2\log \mu} & \text{for} \quad \mu > 1 \end{cases},$$

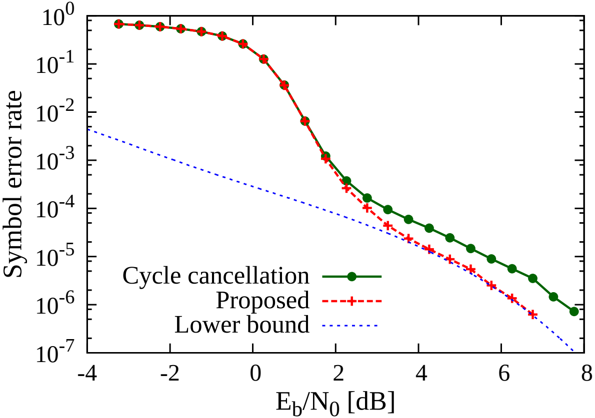
where $Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp[-\frac{x^2}{2}]dx$.

Corollary2 : BSC case

For BSC with crossover probability $\epsilon < \frac{1-\sqrt{1-\max\{\mu^{-2/m},1\}}}{2}$,

$$\mathrm{P_s}(\mathcal{E}) \geq \frac{1}{2N} \sum_{s=s_{\mathrm{c}}}^{\infty} \mu^s \sum_{i \leq ms/2} \binom{ms}{i} (1-\epsilon)^{ms-i} \epsilon^i + o\left(\frac{1}{N}\right).$$

Simulation result : Lower bound (AWGN channel)



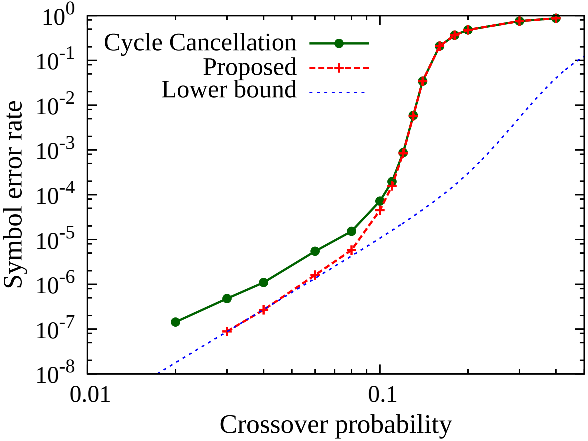
(2,3)-regular LDPC code ensemble

Symbol code length : 315

Order of Galois field : 16

$s_{\mathrm{g}} = 1$

Simulation result : Lower bound (BSC)



(2,3)-regular LDPC code ensemble
Symbol code length : 1200
Order of Galois field : 16
$s_{\mathrm{g}} = 2$

Conclusion and related works

Conclusion

- We derive a necessary and sufficient condition for successful decoding for zigzag cycle codes over the MBIOS channel under BP decoding.
- We propose a design method lowering error floor.
- We give a lower bound for the decoding error rates in the error floors for the expurgated ensembles constructed by proposed method.

Related works

- Analysis of error floors of non-binary LDPC codes over q-ary symmetric channel
- Analysis of error floors of non-binary LDPC codes defined over general linear groups