Analytical Solution of Covariance Evolution for Regular LDPC Codes

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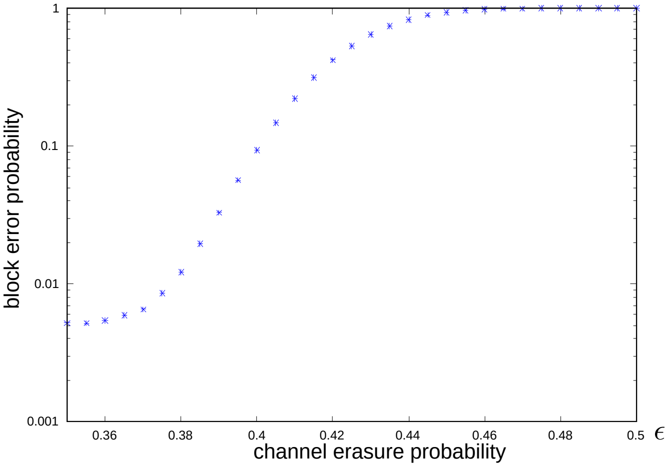
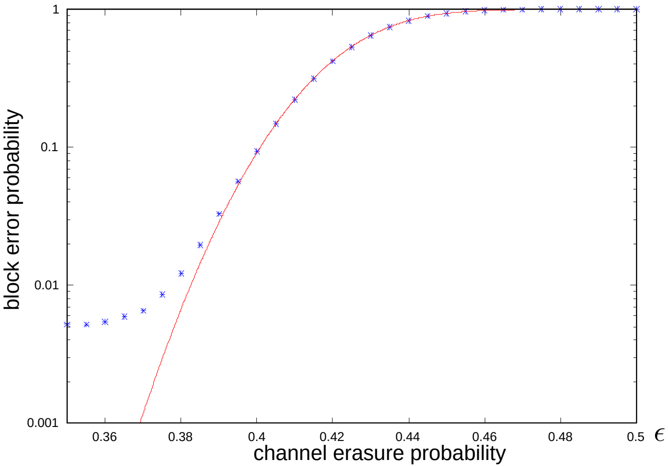
Finite-length scaling [Amraoui, et al. '03]



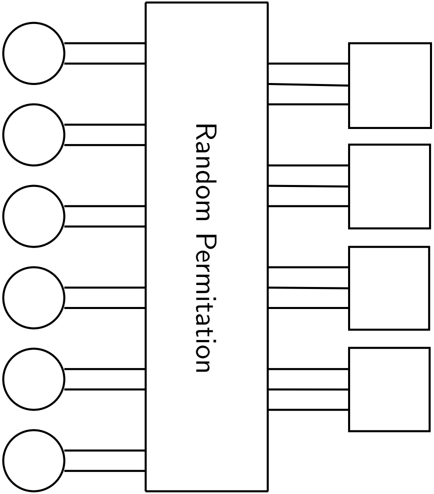
(3,6)-regular, block-length 1024

Finite-length scaling [Amraoui, et al. '03]



(3,6)-regular, block-length 1024

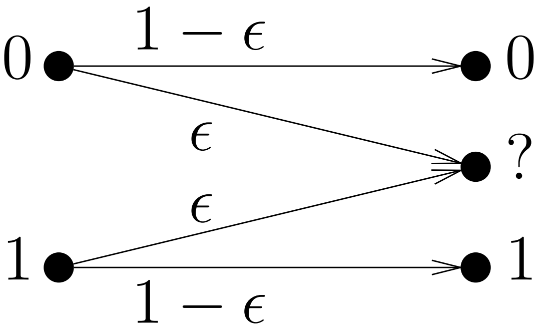
Regular LDPC code ensemble



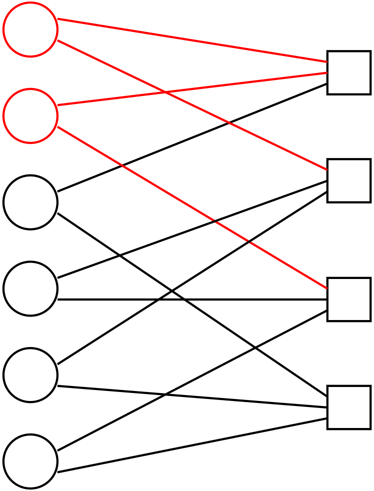
(2,3)-regular LDPC code ensemble

◯ : variable node
□ : check node

Binary Erasure Channel : BEC



Peeling Algorithm [Luby '97]



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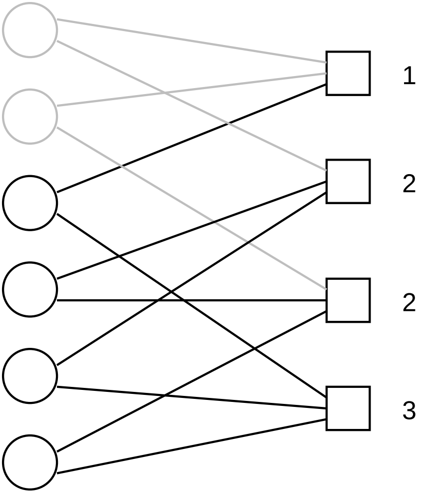

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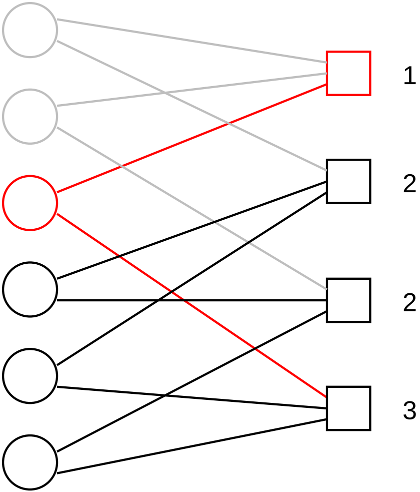

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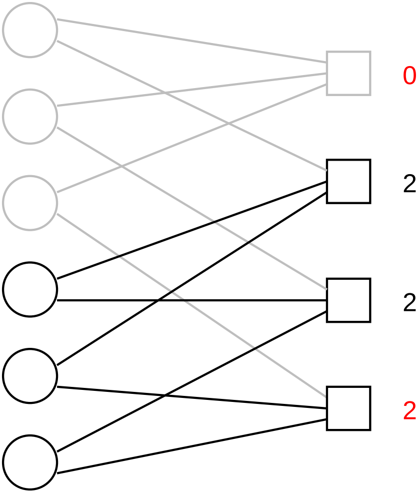
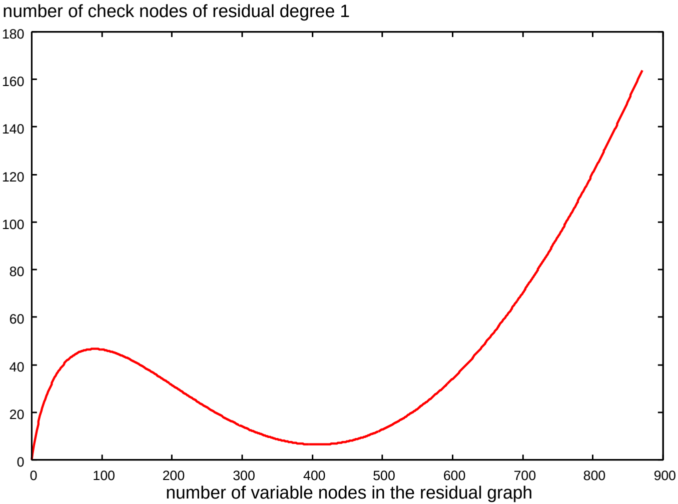


The number of check nodes of residual degree 1



(3,6)-regular, block-length 2048, channel erasure probability 0.425

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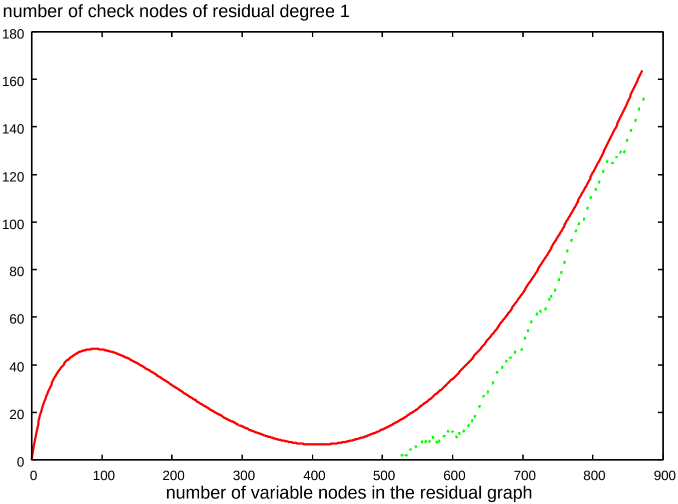

(3,6)-regular, block-length 2048, channel erasure probability 0.425

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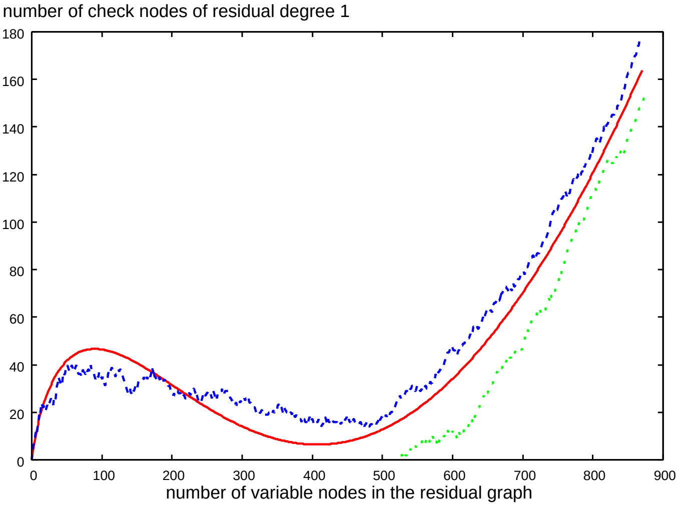
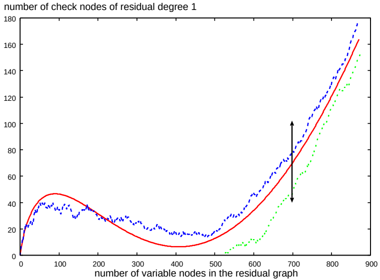


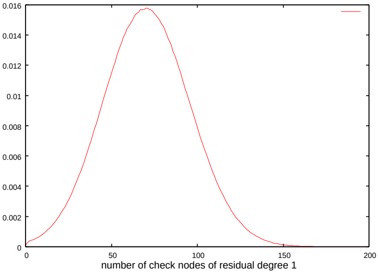
(3,6)-regular, block-length 2048, channel erasure probability 0.425

Distribution of the number of check nodes of residual degree 1 [Amraoui, et al. '03]

Amraoui, et al '03

The distribution of the number of check nodes of residual degree 1 converge weakly to normal distribution.





(3,6)-regular ensemble, block-length 2048, channel erasure probability 0.425

The variance of the number of check node of residual degree 1 [Amraoui, et al. '03]

- numerical computations for covariance evolution
- the variance of the number of erased message

Definition of the covariances

Definition of the covariance

$$\delta^{(l_b,l_b)}(y) := \frac{\mathrm{Cov}(l_b(y), l_b(y))}{\#\ \text{(edges in original graph)}}$$
$$\delta^{(l_b,r_j)}(y) := \frac{\mathrm{Cov}(l_b(y), r_j(y))}{\#\ \text{(edges in original graph)}}$$
$$\delta^{(r_j,r_k)}(y) := \frac{\mathrm{Cov}(r_j(y), r_k(y))}{\#\ \text{(edges in original graph)}}$$

$$l_b(y) := \#\ \text{(edges connecting to variable node of residual degree } b)$$
$$r_j(y) := \#\ \text{(edges connecting to check node of residual degree } j)$$
$$(j \in \{1, 2, \ldots, d-1\})$$
$$y := \text{(parameter determined from the iteration number of PA)}$$

Result

Theorem 1

For (b,d)-regular LDPC code ensemble,

$$\delta^{(l_b,l_b)}(y) = b\epsilon\tilde{\epsilon},$$

$$\delta^{(l_b,r_j)}(y) = -G_j\{\epsilon\tilde{\epsilon}(b-1)y^{-1} + \tilde{\epsilon}x\} + I_{\{j=1\}}b\epsilon\tilde{\epsilon},$$

$$\delta^{(r_k,r_j)}(y) = \frac{b-1}{b}G_kG_j\{\epsilon\tilde{\epsilon}(b-1)y^{-2} - (\epsilon-\tilde{\epsilon})xy^{-1} + x^2\}$$

$$-d\binom{d-1}{k-1}\binom{d-1}{j-1}x^{j+k}\tilde{x}^{2d-k-j} + I_{\{k=j\}}\binom{d-1}{k-1}kx^k\tilde{x}^{d-k}$$

$$+I_{\{k=1,j=1\}}(b\epsilon\tilde{\epsilon} - x\tilde{x}) - (I_{\{k=1\}}G_j + I_{\{j=1\}}G_k)\{\epsilon\tilde{\epsilon}(b-1)y^{-1} - \epsilon x + x^2\}$$

ϵ : channel erasure probability

$x := \epsilon y^{b-1}, \quad \tilde{\epsilon} := 1-\epsilon, \quad \tilde{x} := 1-x,$

$G_j := \binom{d-1}{j-1}x^{j-1}\tilde{x}^{d-j-1}(dx-j) + I_{\{j=1\}}$

Result

$$
\begin{aligned}
\delta^{(r_1,r_1)}(y) =& \frac{b-1}{b} G_1^2\{\epsilon\tilde{\epsilon}(b-1)y^{-2} - (\epsilon-\tilde{\epsilon})xy^{-1} + x^2\} \\
& - dx^2\tilde{x}^{2d-2} + x\tilde{x}^{d-1} \\
& + (b\epsilon\tilde{\epsilon} - x\tilde{x}) - 2G_1\{\epsilon\tilde{\epsilon}(b-1)y^{-1} - \epsilon x + x^2\},
\end{aligned}
$$

where $G_1 := \tilde{x}^{d-j-1}(dx-1)+1$

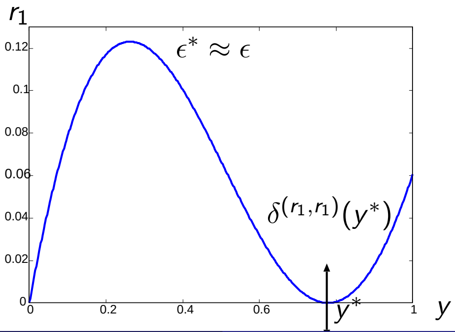
Result

corollary 1

ϵ^* : threshold of the ensemble under BP decoding
for $\epsilon = \epsilon^*$

$$\delta^{(r_1,r_1)}(y^*) = \frac{x^* y^*}{b-1}(y^* - x^*)$$

The same result as in [Amraoui '06] for regular LDPC code ensemble.



y^* : critical point,
$x^* := \epsilon^*(y^*)^{b-1}$

Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

$$\frac{\mathrm{d}\boldsymbol{\delta}}{\mathrm{d}y} = F(y)\boldsymbol{\delta}(y) + \mathbf{b}(y)$$

Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

$$\frac{\mathrm{d}\delta^{(l_b,l_b)}}{\mathrm{d}y}(y) = 0,$$

$$\frac{\mathrm{d}\delta^{(l_b,r_{d-1})}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial \hat{f}^{(r_{d-1})}}{\partial \bar{l}_b}\delta^{(l_b,l_b)} + \sum_{s=1}^{d-1}\frac{\partial \hat{f}^{(r_{d-1})}}{\partial \bar{r}_s}\delta^{(l_b,r_s)}\Big],$$

$$\frac{\mathrm{d}\delta^{(l_b,r_k)}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial \hat{f}^{(r_k)}}{\partial \bar{l}_b}\delta^{(l_b,l_b)} + \frac{\partial \hat{f}^{(r_k)}}{\partial \bar{r}_{k+1}}\delta^{(l_b,r_{k+1})} + \frac{\partial \hat{f}^{(r_k)}}{\partial \bar{r}_k}\delta^{(l_b,r_k)}\Big].$$

Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

$$\delta^{(l_b,l_b)}(y) = b\epsilon\tilde{\epsilon},$$

$$\frac{\mathrm{d}\delta^{(l_b,r_{d-1})}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial \hat{f}^{(r_{d-1})}}{\partial \bar{l}_b} b\epsilon\tilde{\epsilon} + \sum_{s=1}^{d-1} \frac{\partial \hat{f}^{(r_{d-1})}}{\partial \bar{r}_s} \delta^{(l_b,r_s)}\Big],$$

$$\frac{\mathrm{d}\delta^{(l_b,r_k)}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial \hat{f}^{(r_k)}}{\partial \bar{l}_b} b\epsilon\tilde{\epsilon} + \frac{\partial \hat{f}^{(r_k)}}{\partial \bar{r}_{k+1}} \delta^{(l_b,r_{k+1})} + \frac{\partial \hat{f}^{(r_k)}}{\partial \bar{r}_k} \delta^{(l_b,r_k)}\Big].$$

Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

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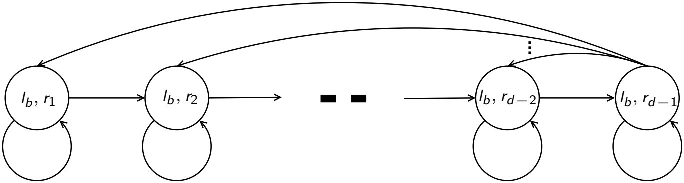

Outline of proof

$A(y) := \sum_{s=1}^{d-1} \delta^{(l_b, r_s)}$

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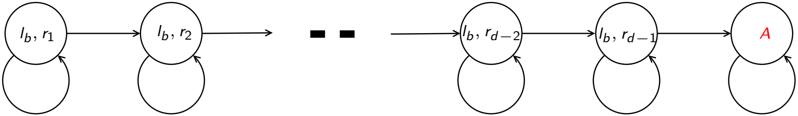
$A(y) := \sum_{s=1}^{d-1} \delta^{(l_b,r_s)}$

$$\frac{\mathrm{d}A}{\mathrm{d}y}(y) = d(b-1)y^{-1}A + d(b-1)(x^{d-1}y^{-2} - y^{-1})b\epsilon\tilde{\epsilon},$$

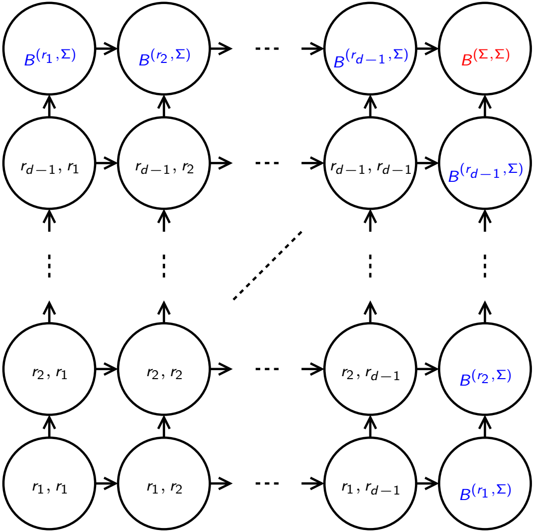
$$\frac{\mathrm{d}\delta^{(l_b,r_{d-1})}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial\hat{f}^{(r_{d-1})}}{\partial\bar{l}_b}b\epsilon\tilde{\epsilon} + \frac{\partial\hat{f}^{(r_{d-1})}}{\partial\bar{r}_1}A + \frac{\partial\hat{f}^{(r_{d-1})}}{\partial\bar{r}_1}\delta^{(l_b,r_{d-1})}\Big],$$

$$\frac{\mathrm{d}\delta^{(l_b,r_k)}}{\mathrm{d}y}(y) = -x\Big[\frac{\partial\hat{f}^{(r_k)}}{\partial\bar{l}_b}b\epsilon\tilde{\epsilon} + \frac{\partial\hat{f}^{(r_k)}}{\partial\bar{r}_{k+1}}\delta^{(l_b,r_{k+1})} + \frac{\partial\hat{f}^{(r_k)}}{\partial\bar{r}_k}\delta^{(l_b,r_k)}\Big].$$

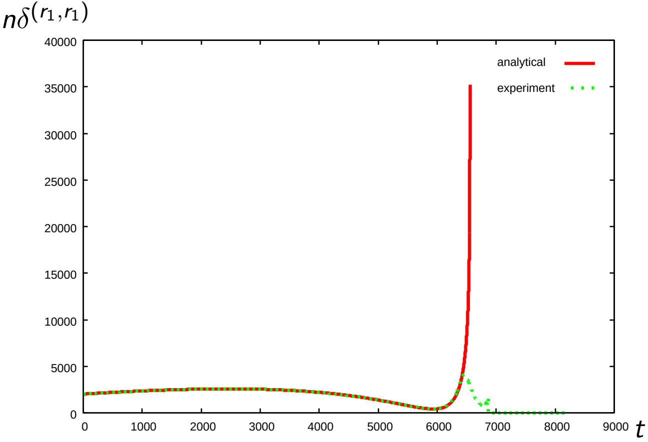


Outline of proof

$B^{(r_j,\Sigma)} := \sum_{s=1}^{d-1} \delta^{(r_j,r_s)}$, $B^{(\Sigma,\Sigma)} := \sum_{j=1}^{d-1} \sum_{s=1}^{d-1} \delta^{(r_j,r_s)}$



numerically computation



(3,6)-regular LDPC code ensemble,
block length 16384, channel erasure probability 0.400

Relationship between stability condition

corollary 2

$$\rho_{l_b,r_1}(\epsilon) = \begin{cases} 1, & \text{if } \; I_{\{b=2\}}(d-1)\epsilon \leq 1 \\ -1, & \text{if } \; I_{\{b=2\}}(d-1)\epsilon > 1. \end{cases}$$

$I_{\{b=2\}}(d-1)\epsilon < 1$: stability condition

where,

$$\rho_{l_b,r_1}(\epsilon) := \lim_{y \to 0} \frac{\delta^{(l_b,r_1)}(\epsilon, y)}{\sqrt{\delta^{(l_b,l_b)}(\epsilon, y)\delta^{(r_1,r_1)}(\epsilon, y)}},$$

Conclusion and Future Work

Conclusion

- We show the analytical solution of the covariance evolution for regular LDPC code ensembles
- We have derived the relation ship between stability condition

Future Work

- Show the analytical solution of the covariance evolution for irregular LDPC code ensembles

Bibliography

- A. Amraoui "Asymptotic and finite-length optimization of LDPC codes," Ph.D. Thesis, EPFL, June 2006.
- M. Luby, M. Mitzenmacher, A. Shokrollahi, D. A. Spielman, and V. Stemann. "Practical Loss-Resilient Codes," in *Proceedings of the 29th annual ACM Symposium on Theory of Computing*, 1997, pp. 150-159.
- T. Richardson and R. Urbanke, *Modern Coding Theory*, Cambridge University Press, 2008.