Analytical Solution of Covariance Evolution for Regular LDPC Codes

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Finite-length scaling [Amraoui, et al. ’03]

(3,6)-regular, block-length 1024
Finite-length scaling  [Amraoui, et al. ’03]

(3,6)-regular, block-length 1024
Regular LDPC code ensemble

(2,3)-regular LDPC code ensemble

○ : variable node
□ : check node
Binary Erasure Channel: BEC

0 \rightarrow 1 - \epsilon \rightarrow 0

\epsilon \rightarrow ? \rightarrow \epsilon

1 \rightarrow 1 - \epsilon \rightarrow 1
Peeling Algorithm [Luby ’97]
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July 3th 2009 7 / 27
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The number of check nodes of residual degree 1

(3,6)-regular, block-length 2048, channel erasure probability 0.425
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The number of check nodes of residual degree 1

(3,6)-regular, block-length 2048, channel erasure probability 0.425
Amraoui, et al. '03

The distribution of the number of check nodes of residual degree 1 converge weakly to normal distribution.

(3,6)-regular ensemble, block-length 2048, channel erasure probability 0.425
The variance of the number of check node of residual degree 1 [Amraoui, et al. ’03]

- numerical computations for covariance evolution
- the variance of the number of erased message
Definition of the covariances

**Definition of the covariance**

\[
\begin{align*}
\delta(l_b, l_b)(y) & := \frac{\text{Cov}(l_b(y), l_b(y))}{\# \text{ (edges in original graph)}} \\
\delta(l_b, r_j)(y) & := \frac{\text{Cov}(l_b(y), r_j(y))}{\# \text{ (edges in original graph)}} \\
\delta(r_j, r_k)(y) & := \frac{\text{Cov}(r_j(y), r_k(y))}{\# \text{ (edges in original graph)}}
\end{align*}
\]

\[
l_b(y) := \# \text{ (edges connecting to variable node of residual degree } b)\\
r_j(y) := \# \text{ (edges connecting to check node of residual degree } j)\\
(j \in \{1, 2, \ldots, d - 1\})\\
y := \text{(parameter determined from the iteration number of PA)}
\]
Theorem 1

For \((b, d)\)-regular LDPC code ensemble,

\[
\delta(l_b; l_b)(y) = b\epsilon\tilde{\epsilon},
\]

\[
\delta(l_b; r_j)(y) = -G_j\{\epsilon\tilde{\epsilon}(b - 1)y^{-1} + \tilde{\epsilon}x\} + I_{\{j=1\}} b\epsilon\tilde{\epsilon},
\]

\[
\delta(r_k; r_j)(y) = \frac{b - 1}{b} G_k G_j\{\epsilon\tilde{\epsilon}(b - 1)y^{-2} - (\epsilon - \tilde{\epsilon})xy^{-1} + x^2\}
- d \binom{d - 1}{k - 1} \binom{d - 1}{j - 1} x^{j+k}\tilde{x}^{2d-k-j} + I_{\{k=j\}} \binom{d - 1}{k - 1} k x^k \tilde{x}^{d-k}
+ I_{\{k=1, j=1\}} (b\epsilon\tilde{\epsilon} - x\tilde{x}) - (I_{\{k=1\}} G_j + I_{\{j=1\}} G_k)\{\epsilon\tilde{\epsilon}(b - 1)y^{-1} - \epsilon x + x^2\}
\]

\(\epsilon\) : channel erasure probability
\(x := \epsilon y^{b-1}\), \(\tilde{\epsilon} := 1 - \epsilon\), \(\tilde{x} := 1 - x\),
\(G_j := \binom{d-1}{j-1} x^{j-1}\tilde{x}^{d-j-1}(dx - j) + I_{\{j=1\}}\)
\[
\delta^{(r_1,r_1)}(y) = \frac{b - 1}{b} G_1^2 \{ \epsilon \tilde{\epsilon} (b - 1)y^{-2} - (\epsilon - \tilde{\epsilon})xy^{-1} + x^2 \} \\
- dx^2 \tilde{x}^{2d-2} + x\tilde{x}^{d-1} \\
+ (b\epsilon \tilde{\epsilon} - x\tilde{x}) - 2G_1 \{ \epsilon \tilde{\epsilon} (b - 1)y^{-1} - \epsilon x + x^2 \},
\]

where \( G_1 := \tilde{x}^{d-j-1}(dx - 1) + 1 \).
corollary 1

\( \epsilon^* \) : threshold of the ensemble under BP decoding for \( \epsilon = \epsilon^* \)

\[
\delta(r_1, r_1)(y^*) = \frac{x^*y^*}{b-1}(y^* - x^*)
\]

The same result as in [Amraoui '06] for regular LDPC code ensemble.

\( y^* \) : critical point, 
\( x^* := \epsilon^*(y^*)^{b-1} \)
Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui ’06])

\[ \frac{d\delta}{dy} = F(y)\delta(y) + b(y) \]
Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

\[
\frac{d \delta(l_b,l_b)}{dy} = 0,
\]

\[
\frac{d \delta(l_b,r_{d-1})}{dy} = -x \left[ \frac{\partial \hat{f}(r_{d-1})}{\partial l_b} \delta(l_b,l_b) + \sum_{s=1}^{d-1} \frac{\partial \hat{f}(r_{d-1})}{\partial r_s} \delta(l_b,r_s) \right],
\]

\[
\frac{d \delta(l_b,r_k)}{dy} = -x \left[ \frac{\partial \hat{f}(r_k)}{\partial l_b} \delta(l_b,l_b) + \frac{\partial \hat{f}(r_k)}{\partial r_{k+1}} \delta(l_b,r_{k+1}) + \frac{\partial \hat{f}(r_k)}{\partial r_k} \delta(l_b,r_k) \right].
\]
Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

\[
\delta(l_b,l_b)(y) = b\epsilon\tilde{\epsilon},
\]

\[
\frac{d\delta(l_b,r_{d-1})}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_{d-1})}{\partial l_b} b\epsilon\tilde{\epsilon} + \sum_{s=1}^{d-1} \frac{\partial \hat{f}(r_{d-1})}{\partial r_s} \delta(l_b,r_s) \right],
\]

\[
\frac{d\delta(l_b,r_k)}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_k)}{\partial l_b} b\epsilon\tilde{\epsilon} + \frac{\partial \hat{f}(r_k)}{\partial r_{k+1}} \delta(l_b,r_{k+1}) + \frac{\partial \hat{f}(r_k)}{\partial r_k} \delta(l_b,r_k) \right].
\]
Outline of proof

Solve the system of differential equation (covariance evolution) [Amraoui '06])

\[ \delta(l_b, l_b)(y) = b \epsilon \tilde{\epsilon}, \]

\[ \frac{d \delta(l_b, r_{d-1})}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_{d-1})}{\partial l_b} b \epsilon \tilde{\epsilon} + \sum_{s=1}^{d-1} \frac{\partial \hat{f}(r_{d-1})}{\partial r_s} \delta(l_b, r_s) \right], \]

\[ \frac{d \delta(l_b, r_k)}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_k)}{\partial l_b} b \epsilon \tilde{\epsilon} + \frac{\partial \hat{f}(r_k)}{\partial r_{k+1}} \delta(l_b, r_{k+1}) + \frac{\partial \hat{f}(r_k)}{\partial r_k} \delta(l_b, r_k) \right]. \]
Outline of proof

\[ A(y) := \sum_{s=1}^{d-1} \delta(l_b, r_s) \]
Outline of proof

\[ A(y) := \sum_{s=1}^{d-1} \delta(l_b, r_s) \]

\[ \frac{dA}{dy}(y) = d(b - 1)y^{-1}A + d(b - 1)(x^{d-1}y^{-2} - y^{-1})b\epsilon\bar{\epsilon}, \]

\[ \frac{d\delta(l_b, r_{d-1})}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_{d-1})}{\partial l_b} b\epsilon\bar{\epsilon} + \frac{\partial \hat{f}(r_{d-1})}{\partial r_1} A + \frac{\partial \hat{f}(r_{d-1})}{\partial r_1} \delta(l_b, r_{d-1}) \right], \]

\[ \frac{d\delta(l_b, r_k)}{dy}(y) = -x \left[ \frac{\partial \hat{f}(r_k)}{\partial l_b} b\epsilon\bar{\epsilon} + \frac{\partial \hat{f}(r_k)}{\partial r_{k+1}} \delta(l_b, r_{k+1}) + \frac{\partial \hat{f}(r_k)}{\partial r_k} \delta(l_b, r_k) \right]. \]
Outline of proof

\[ B^{(r_j, \Sigma)} := \sum_{s=1}^{d-1} \delta(r_j, r_s), \quad B^{(\Sigma, \Sigma)} := \sum_{j=1}^{d-1} \sum_{s=1}^{d-1} \delta(r_j, r_s) \]
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Analytical Solution of Covariance Evolution for Regular LDPC Codes

July 3th 2009 24 / 27

$\eta_\delta(r_1,r_1)$

(3,6)-regular LDPC code ensemble,
block length 16384, channel erasure probability 0.400
Relationship between stability condition

corollary 2

\[ \rho_{l_{b}, r_{1}}(\epsilon) = \begin{cases} 1, & \text{if } l_{\{b=2\}}(d - 1)\epsilon \leq 1 \\ -1, & \text{if } l_{\{b=2\}}(d - 1)\epsilon > 1. \end{cases} \]

\( l_{\{b=2\}}(d - 1)\epsilon < 1 : \text{stability condition} \)

where,

\[ \rho_{l_{b}, r_{1}}(\epsilon) := \lim_{y \to 0} \frac{\delta(l_{b}, r_{1})(\epsilon, y)}{\sqrt{\delta(l_{b}, l_{b})(\epsilon, y)\delta(r_{1}, r_{1})(\epsilon, y)}}, \]
Conclusion

- We show the analytical solution of the covariance evolution for regular LDPC code ensembles
- We have derived the relation ship between stability condition

Future Work

- Show the analytical solution of the covariance evolution for irregular LDPC code ensembles
