Error Floors of Non-binary LDPC Codes

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Outline

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  - Existing works
- Lowering the error floor
  - Condition for successful decoding under BP decoding
  - Modified cycle cancellation
  - Simulation results
- Analysis of the error floors
  - Closed-form expression for error floor
  - Monotonicity of error floors

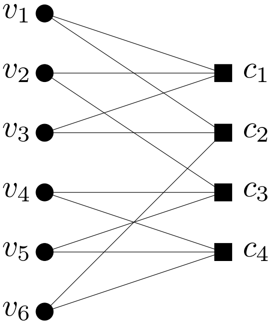
Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

Linear code defined by a sparse parity check matrix $H \in \mathbb{F}_{2^m}^{M \times N}$

$$C := \{\mathbf{x} \in \mathbb{F}_{2^m}^{N} \mid H\mathbf{x} = \mathbf{0}\}$$

- It is empirically known that $(2,k)$-regular LDPC codes exhibit good decoding performance.



$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\ 0 & h_{3,2} & 0 & h_{3,4} & h_{3,5} & 0 \\ 0 & 0 & 0 & h_{4,4} & h_{4,5} & h_{4,6} \end{pmatrix} \end{array}$$

Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

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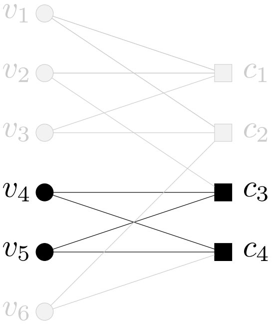

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Zigzag cycles degrade decoding performance

- Error floors are mainly caused by codewords of small weight.

Example : Zigzag cycle yields codeword

$$H = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\ 0 & h_{3,2} & 0 & 1 & \alpha^4 & 0 \\ 0 & 0 & 0 & \alpha^6 & \alpha^{10} & h_{4,6} \end{pmatrix}$$

$$(0 \;\; 0 \;\; 0 \;\; \alpha^4 \;\; 1 \;\; 0) \in C$$

$$\det \begin{pmatrix} 1 & \alpha^4 \\ \alpha^6 & \alpha^{10} \end{pmatrix} = 0$$

Zigzag cycle yields codeword.

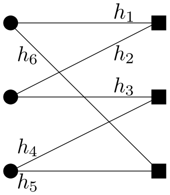
Cycle cancellation [Poulliat 2008]

Existing work : Cycle cancellation [Poulliat 2008]

Design edge labels in zigzag cycle so that the corresponding sub-matrices $\tilde{H}$ are nonsingular, i.e.,

$$\det \tilde{H} \neq 0.$$

$$\tilde{H} = \begin{pmatrix} h_1 & h_2 & 0 \\ 0 & h_3 & h_4 \\ h_6 & 0 & h_5 \end{pmatrix}$$



$$\det \tilde{H} = h_1 h_3 h_5 + h_2 h_4 h_6 \neq 0 \iff \beta = h_1 h_2^{-1} h_3 h_4^{-1} h_5 h_6^{-1} \neq 1$$

$$\iff \beta = \prod_{i=1}^{3} h_{2i-1} h_{2i}^{-1} \neq 1$$

[Poulliat 2008] C. Poulliat, M. Fossorier and D. Declercq, "Design of regular (2, d_c)-LDPC codes over GF(q) using their binary images", IEEE Trans. Comm. (2008)

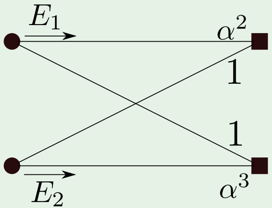
Motivation

Some zigzag cycles designed by cycle cancellation can cause decoding failure under BP decoding over the BEC.

Example : Decoding failure under BP decoding

Zigzag cycle $N = 2$, $\mathbb{F}_{2^4}$

$$E_1 = \{0, \alpha^0, \alpha^5, \alpha^{10}\}$$
$$E_2 = \{0, \alpha^2, \alpha^7, \alpha^{12}\}$$



Note that for the BEC

- All the nonzero entries in a message are equal
- Messages are represented by the set of indexes of nonzero entries

Condition for successful decoding for zigzag cycle codes

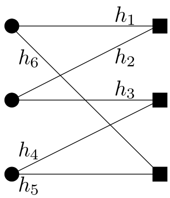
Theorem 1

A necessary condition for successful decoding under BP decoding for zigzag cycle code over the BEC is that

1. not all bits are erased
2. cycle parameter β is not in the proper subfield of $\mathbb{F}_{2^m}$, i.e.,

$$\beta \notin \mathcal{H}_m := \bigcup_{r>0:r|m,r\neq m} \{\alpha^{i\frac{2^m-1}{2^r-1}} \mid i = 0, 1, \ldots, 2^r - 2\}$$

α : primitive element of $\mathbb{F}_{2^m}$



cycle parameter
$\beta = \prod_{i=1}^{n} h_{2i-1} h_{2i}^{-1}$

Modified cycle cancellation

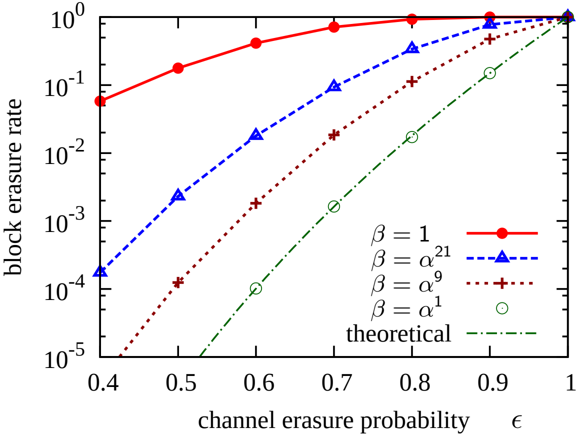
Modified cycle cancellation

Design the edge labels in zigzag cycle such that

$$\beta \notin \mathcal{H}_m.$$

Field	The elements of $\mathcal{H}_m$
$\mathbb{F}_{2^4}$	$1, \alpha^5, \alpha^{10}$
$\mathbb{F}_{2^5}$	1
$\mathbb{F}_{2^6}$	$1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}$
$\mathbb{F}_{2^7}$	1
$\mathbb{F}_{2^8}$	$1, \alpha^{17}, \alpha^{34}, \alpha^{51}, \alpha^{68}, \alpha^{85}, \alpha^{102}, \alpha^{119}, \alpha^{136}, \alpha^{153}, \alpha^{170}, \alpha^{187}, \alpha^{204}, \alpha^{221}, \alpha^{238}$
$\mathbb{F}_{2^9}$	$1, \alpha^{73}, \alpha^{146}, \alpha^{219}, \alpha^{292}, \alpha^{365}, \alpha^{438}$

Simulation result : Zigzag cycle codes (1)

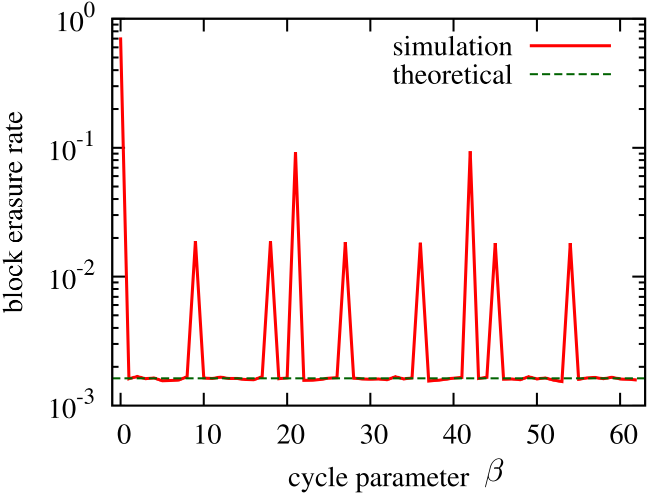


Zigzag cycle code, $N = 3$ (18 bits), $\mathbb{F}_{2^6}$

$$\mathcal{H}_6 = \{1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}\}$$

The theoretical block erasure rate is given by ϵ^{18}.

Simulation result : Zigzag cycle codes (2)

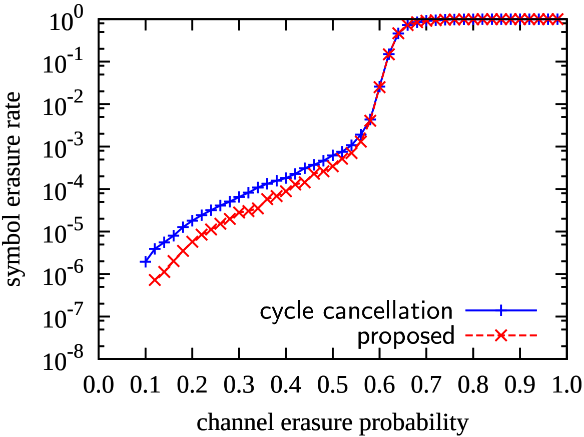


Zigzag cycle code $N = 3$ (18 bits), $\mathbb{F}_{2^6}$, $\epsilon = 0.7$

$$\mathcal{H}_6 = \{1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}\}$$

The theoretical block erasure rate is given by $0.7^{18} \approx 1.63 \times 10^{-3}$.

Simulation result : LDPC codes



(2,3)-regular LDPC codes

Symbol code length : 315 symbols ($315 \times 4 = 1260$ bits)

Order of Galois field : $16 = 2^4$

Closed-form expression of error floors

Definition : LDPC code ensemble

Let $\mathrm{LDPC}(N, m, \lambda, \rho)$ denote the set of LDPC codes of symbol code length N over $\mathbb{F}_{2^m}$ defined by Tanner graphs with a degree distribution pair (λ, ρ).

Definition : Expurgate ensemble

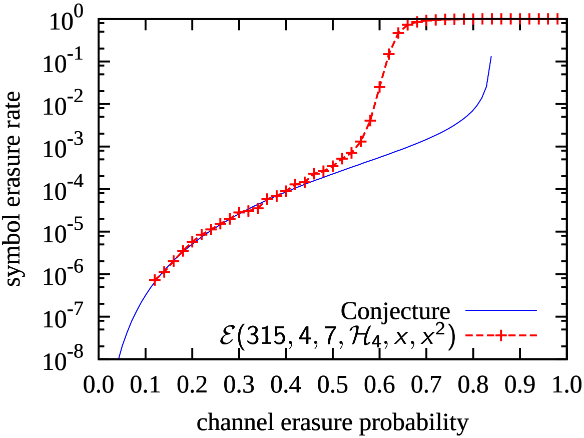
Let $\mathcal{E}(N, m, s_c, \mathcal{H}, \lambda, \rho)$ be the set of codes in $\mathrm{LDPC}(N, m, \lambda, \rho)$ which contains no zigzag cycles of size at most s_c with cycle parameter $\beta \in \mathcal{H}$.

Conjecture 1 : Error floor for NB-LDPC codes

Let $\mathrm{P_s}(N, s_c, \epsilon)$ be the symbol erasure rate of $\mathcal{E}(N, m, s_c, \mathcal{H}_m, \lambda, \rho)$ over BEC(ϵ). Then,

$$\lim_{s_c \to \infty} \lim_{N \to \infty} N\mathrm{P_s}(N, s_c, \epsilon) = \frac{1}{2} \frac{\lambda'(0)\rho'(1)\epsilon^m}{1 - \lambda'(0)\rho'(1)\epsilon^m}$$

Simulation result : Error floor



(2,3)-regular LDPC code ensemble designed by our proposed method

Symbol code length : 315 symbols ($315 \times 4 = 1260$ bits)

Order of Galois filed : $16 = 2^4$

Monotonicity of error floor

Conjecture 2 : Monotonicity of error floor

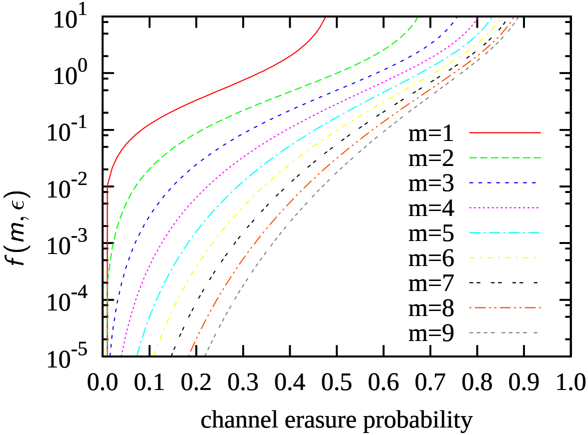
Let $n := mN$ denote bit code length.
Let $\mathrm{P_b}(n, m, s_c, \epsilon)$ be the bit erasure rate of $\mathcal{E}(N, m, s_c, \mathcal{H}_m, \lambda, \rho)$ over BEC(ϵ).

$$
\begin{aligned}
f(m,\epsilon) :=& \lim_{s_c\to\infty} \lim_{n\to\infty} n\mathrm{P_b}(n, m, s_c, \epsilon) \\
=& \frac{m}{2} \frac{\lambda'(0)\rho'(1)\epsilon^m}{1-\lambda'(0)\rho'(1)\epsilon^m} \qquad \text{(From Conjecture 1)}
\end{aligned}
$$

For $0 < \epsilon < (\lambda'(0)\rho'(1))^{-\frac{1}{m}}$ and $\lambda'(0)\rho'(1) > 1$,

$$
f(m,\epsilon) > f(m+1,\epsilon).
$$

Monotonicity of error floor



(2,3)-regular LDPC codes ensemble constructed by our proposed method

Conclusion and future work

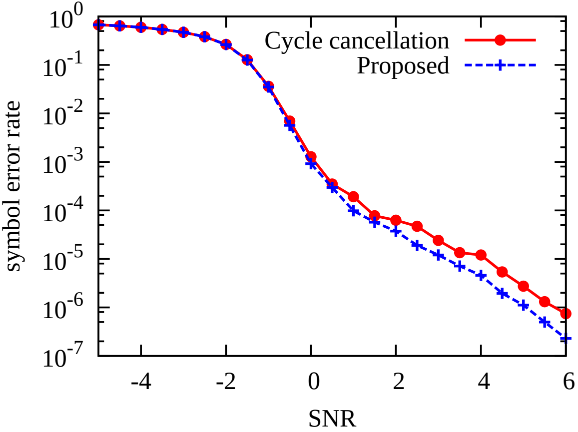
Conclusion

- We derive a necessary condition for successful decoding for zigzag cycle codes over the BEC under BP decoding.
- We propose a design method lowering error floor.
- We analyze the error floors for the expurgated ensembles constructed by proposed method.
- We show that error floors decrease as the size of Galois field increases.

Future work

- Clarify the condition for successful decoding for 3 imbricate cycles.

Simulation result : LDPC codes over AWGNC



(2,3)-regular LDPC codes ensemble over AWGNC

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