Error Floors of Non-binary LDPC Codes

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2010/6/14
Outline

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  - Existing works
- Lowering the error floor
  - Condition for successful decoding under BP decoding
  - Modified cycle cancellation
  - Simulation results
- Analysis of the error floors
  - Closed-form expression for error floor
  - Monotonicity of error floors
Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

Linear code defined by a sparse parity check matrix $H \in \mathbb{F}_{2^m}^{M \times N}$

$$C := \{ \mathbf{x} \in \mathbb{F}_{2^m}^N \mid H\mathbf{x} = \mathbf{0} \}$$

- It is empirically known that $(2,k)$-regular LDPC codes exhibit good decoding performance.

\[
\begin{align*}
v_1 & \quad c_1 \\
v_2 & \quad c_1 \\
v_3 & \quad c_2 \\
v_4 & \quad c_3 \\
v_5 & \quad c_4 \\
v_6 & \\
\end{align*}
\]

\[
\begin{pmatrix}
c_1 & \left( \begin{array}{ccccccc}
h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\
h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\
0 & h_{3,2} & 0 & h_{3,4} & h_{3,5} & 0 \\
0 & 0 & 0 & h_{4,4} & h_{4,5} & h_{4,6} \\
\end{array} \right) \\
c_2 & \end{pmatrix}
\]
Non-binary low-density parity-check (LDPC) codes

Non-binary LDPC code

Linear code defined by a sparse parity check matrix $H \in \mathbb{F}_{2^m}^{M \times N}$

$$C := \{ \mathbf{x} \in \mathbb{F}_{2^m}^N \mid H\mathbf{x} = \mathbf{0} \}$$

- It is empirically known that $(2,k)$-regular LDPC codes exhibit good decoding performance.
Zigzag cycles degrade decoding performance

- Error floors are mainly caused by codewords of small weight.

Example: Zigzag cycle yields codeword

\[ H = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ h_{2,1} & 0 & h_{2,3} & 0 & 0 & h_{2,6} \\ 0 & h_{3,2} & 0 & 1 & \alpha^4 & 0 \\ 0 & 0 & 0 & \alpha^6 & \alpha^{10} & h_{4,6} \end{pmatrix} \]

\[ (0 \ 0 \ 0 \ \alpha^4 \ 1 \ 0) \in C \]

\[ \det \begin{pmatrix} 1 & \alpha^4 \\ \alpha^6 & \alpha^{10} \end{pmatrix} = 0 \]

Zigzag cycle yields codeword.
Cycle cancellation [Poulliat 2008]

Design edge labels in zigzag cycle so that the corresponding sub-matrices $\tilde{H}$ are nonsingular, i.e., $\det \tilde{H} \neq 0$.

$$\tilde{H} = \begin{pmatrix} h_1 & h_2 & 0 \\ 0 & h_3 & h_4 \\ h_6 & 0 & h_5 \end{pmatrix}$$

$$\det \tilde{H} = h_1 h_3 h_5 + h_2 h_4 h_6 \neq 0 \iff \beta = h_1 h_2^{-1} h_3 h_4^{-1} h_5 h_6^{-1} \neq 1$$

$$\iff \beta = \prod_{i=1}^{3} h_{2i-1}^{-1} h_2 h_{2i}^{-1} \neq 1$$

Motivation

Some zigzag cycles designed by cycle cancellation can cause decoding failure under BP decoding over the BEC.

Example: Decoding failure under BP decoding

Zigzag cycle $N = 2$, $\mathbb{F}_2$

$E_1 = \{0, \alpha^0, \alpha^5, \alpha^{10}\}$

$E_2 = \{0, \alpha^2, \alpha^7, \alpha^{12}\}$

Note that for the BEC

- All the nonzero entries in a message are equal
- Messages are represented by the set of indexes of nonzero entries
Theorem 1

A necessary condition for successful decoding under BP decoding for zigzag cycle code over the BEC is that

1. not all bits are erased
2. cycle parameter $\beta$ is not in the proper subfield of $\mathbb{F}_{2^m}$, i.e.,

$$\beta \notin \mathcal{H}_m := \bigcup_{r > 0: r | m, r \neq m} \{ \alpha^{2^r - 1} | i = 0, 1, \ldots, 2^r - 2 \}$$

$\alpha$ : primitive element of $\mathbb{F}_{2^m}$

The cycle parameter

$$\beta = \prod_{i=1}^{n} h_{2i-1} h_{2i}^{-1}$$
Modified cycle cancellation

Design the edge labels in zigzag cycle such that

$$\beta \notin \mathcal{H}_m.$$ 

<table>
<thead>
<tr>
<th>Field</th>
<th>The elements of $\mathcal{H}_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{F}_{2^4}$</td>
<td>$1, \alpha^5, \alpha^{10}$</td>
</tr>
<tr>
<td>$\mathbb{F}_{2^5}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\mathbb{F}_{2^6}$</td>
<td>$1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}$</td>
</tr>
<tr>
<td>$\mathbb{F}_{2^7}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\mathbb{F}_{2^8}$</td>
<td>$1, \alpha^{17}, \alpha^{34}, \alpha^{51}, \alpha^{68}, \alpha^{85}, \alpha^{102}, \alpha^{119}, \alpha^{136}, \alpha^{153}, \alpha^{170}, \alpha^{187}, \alpha^{204}, \alpha^{221}, \alpha^{238}$</td>
</tr>
<tr>
<td>$\mathbb{F}_{2^9}$</td>
<td>$1, \alpha^{73}, \alpha^{146}, \alpha^{219}, \alpha^{292}, \alpha^{365}, \alpha^{438}$</td>
</tr>
</tbody>
</table>
Simulation result: Zigzag cycle codes (1)

Zigzag cycle code, \( N = 3 \) (18 bits), \( \mathbb{F}_{2^6} \)

\[ \mathcal{H}_6 = \{1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}\} \]

The theoretical block erasure rate is given by \( \epsilon^{18} \).
Zigzag cycle code $N = 3$ (18 bits), $\mathbb{F}_{2^6}$, $\epsilon = 0.7$

$\mathcal{H}_6 = \{1, \alpha^9, \alpha^{18}, \alpha^{21}, \alpha^{27}, \alpha^{36}, \alpha^{42}, \alpha^{45}, \alpha^{54}\}$

The theoretical block erasure rate is given by $0.7^{18} \approx 1.63 \times 10^{-3}$. 
Simulation result: LDPC codes

(2,3)-regular LDPC codes
Symbol code length: 315 symbols (315 × 4 = 1260 bits)
Order of Galois field: $16 = 2^4$
Closed-form expression of error floors

**Definition : LDPC code ensemble**

Let $\text{LDPC}(N, m, \lambda, \rho)$ denote the set of LDPC codes of symbol code length $N$ over $\mathbb{F}_{2^m}$ defined by Tanner graphs with a degree distribution pair $(\lambda, \rho)$.

**Definition : Expurgate ensemble**

Let $\mathcal{E}(N, m, s_c, \mathcal{H}, \lambda, \rho)$ be the set of codes in $\text{LDPC}(N, m, \lambda, \rho)$ which contains no zigzag cycles of size at most $s_c$ with cycle parameter $\beta \in \mathcal{H}$.

**Conjecture 1 : Error floor for NB-LDPC codes**

Let $P_s(N, s_c, \epsilon)$ be the symbol erasure rate of $\mathcal{E}(N, m, s_c, \mathcal{H}_m, \lambda, \rho)$ over BEC($\epsilon$). Then,

$$\lim_{s_c \to \infty} \lim_{N \to \infty} NP_s(N, s_c, \epsilon) = \frac{1}{2} \frac{\lambda'(0) \rho'(1) \epsilon^m}{1 - \lambda'(0) \rho'(1) \epsilon^m}$$
Simulation result: Error floor

symbol erasure rate vs. channel erasure probability

Conjecture \( E(315, 4, 7, \mathcal{H}_4, x, x^2) \)

\((2,3)\)-regular LDPC code ensemble designed by our proposed method

Symbol code length: 315 symbols \((315 \times 4 = 1260 \text{ bits})\)
Order of Galois field: \(16 = 2^4\)
Conjecture 2: Monotonicity of error floor

Let \( n := mN \) denote bit code length.
Let \( P_b(n, m, s_c, \epsilon) \) be the bit erasure rate of \( \mathcal{E}(N, m, s_c, \mathcal{H}_m, \lambda, \rho) \) over \( \text{BEC}(\epsilon) \).

\[
\begin{align*}
  f(m, \epsilon) &:= \lim_{s_c \to \infty} \lim_{n \to \infty} nP_b(n, m, s_c, \epsilon) \\
  &= \frac{m}{2} \frac{\lambda'(0)\rho'(1)\epsilon^m}{1 - \lambda'(0)\rho'(1)\epsilon^m} \\
  &\quad \text{(From Conjecture 1)}
\end{align*}
\]

For \( 0 < \epsilon < (\lambda'(0)\rho'(1))^{-\frac{1}{m}} \) and \( \lambda'(0)\rho'(1) > 1 \),

\[ f(m, \epsilon) > f(m + 1, \epsilon). \]
(2,3)-regular LDPC codes ensemble constructed by our proposed method
## Conclusion

- We derive a necessary condition for successful decoding for zigzag cycle codes over the BEC under BP decoding.
- We propose a design method lowering error floor.
- We analyze the error floors for the expurgated ensembles constructed by proposed method.
- We show that error floors decrease as the size of Galois field increases.

## Future work

- Clarify the condition for successful decoding for 3 imbricate cycles.
Simulation result : LDPC codes over AWGNC

(2,3)-regular LDPC codes ensemble over AWGNC

symbol code length : 315 symbols
Order of Galois field : $2^4$