Analysis of Stopping Constellation Distribution for Irregular Non-Binary LDPC Code Ensemble

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Background

stopping constellation

Stopping constellations are the fixed points of BP decoder for non-binary LDPC codes

Relative works

	Weight distribution	Stopping set dist. Stopping constellation dist.
Binary irregular	Di et al. (2006)	Orlitsky et al. (2005)
NB regular	Kasai et al. (2008)	This research
NB irregular	Kasai et al. (2008)	This research

Outline

Non-binary low-density parity-check (LDPC) code

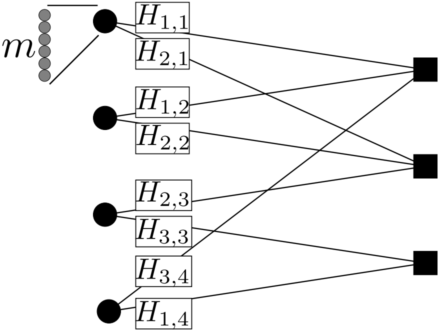
[Definition] Non-binary LDPC code

$$\mathcal{C} := \left\{ (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) \in \mathbb{F}_2^{mN} \mid \forall \mathsf{c} \in \mathtt{C} \ \sum_{i \in \mathcal{N}(\mathsf{c})} H_{i,\mathsf{c}} \boldsymbol{x}_i^T = \mathbf{0}^T \right\}$$

$\mathtt{C}$: Set of the check nodes

$\mathcal{N}(\mathsf{c})$: Set of the variable nodes connecting to the node c

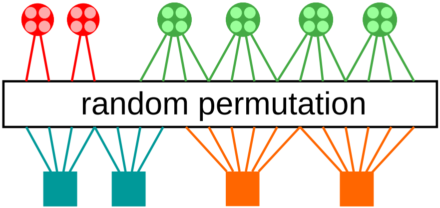
$H_{i,\mathsf{c}}$: Non-singular $m \times m$-matrix (i.e., element of general linear group)



$$\begin{pmatrix} H_{1,1} & H_{1,2} & 0 & H_{1,4} \\ H_{2,1} & H_{2,2} & H_{2,3} & 0 \\ 0 & 0 & H_{3,3} & H_{3,4} \end{pmatrix}$$

Non-binary irregular LDPC code ensemble

$\mathrm{EGL}(N, \lambda, \rho, m)$



$N = 6$, $m = 4$, $\xi = 20$, $r = \frac{1}{3}$

$\lambda_2 = \frac{4}{20}$, $\lambda_4 = \frac{16}{20}$, $\rho_4 = \frac{8}{20}$, $\rho_6 = \frac{12}{20}$

$L_2 = \frac{2}{6}$, $L_4 = \frac{4}{6}$, $R_4 = \frac{2}{4}$, $R_6 = \frac{2}{4}$

λ_i : Fraction of edges connecting to variable node of degree i
ρ_j : Fraction of edges connecting to check node of degree j
L_i : Fraction of variable nodes of degree i
R_j : Fraction of check node of degree j
N : Symbol code length
ξ : Total number of edges
r : Design rate

Non-binary peeling algorithm [Rathi 2008]

[Definition] State

E_{v} : Set of candidate symbols for the decoding result for variable node v

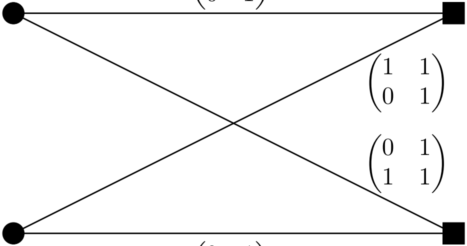
[Property]: E_{v} is a linear subspace of $\mathbb{F}_2^m$ if all-zero codewords are sent.
All-zero assumption holds for non-binary LDPC code over the BEC.

Non-binary peeling algorithm

- Decoding proceeds as the dimensions of the states decrease

[Rathi 2008] V. Rathi, "Conditional entropy of non-binary LDPC codes over the BEC," in proc. ISIT2008, July 2008.

Peeling decoder



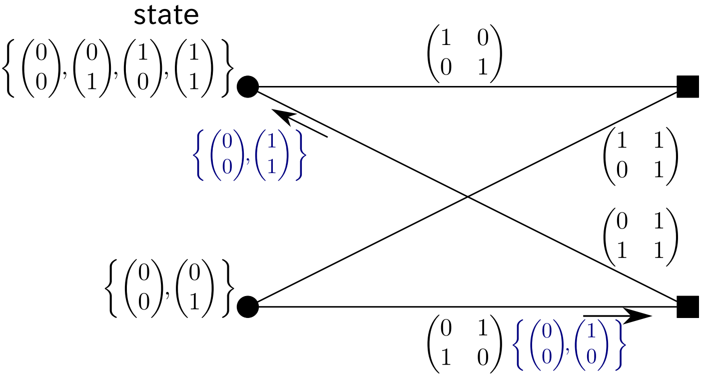
$$\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\begin{pmatrix}0&1\\1&1\end{pmatrix}\quad\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

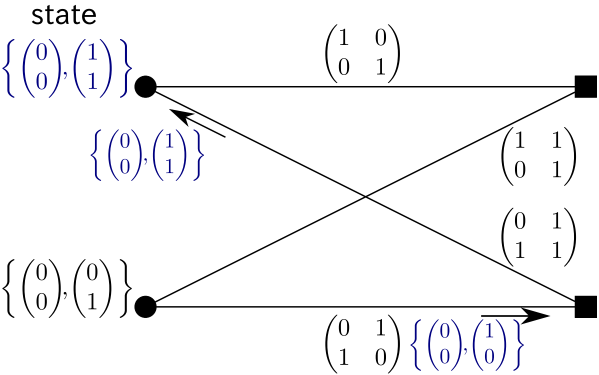
$$m = 2$$

Peeling decoder



$m = 2$

Peeling decoder



$m = 2$

Stopping constellation [Rathi 2008]

[Definition] Stopping constellation

Stopping constellation $\{E_{\mathtt{v}}\}_{\mathtt{v}\in\mathtt{V}}$

$$E_{\mathtt{v}} \subseteq H_{\mathtt{c},\mathtt{v}}^{-1}\left(\sum_{i\in\mathcal{N}(\mathtt{c})\setminus\{\mathtt{v}\}} H_{\mathtt{c},i}E_i\right) \quad \forall \mathtt{v}\in\mathtt{V}, \forall \mathtt{c}\in\mathcal{N}(\mathtt{v})$$

E_i : State for the i-th variable node
$H_{\mathtt{c},\mathtt{v}}$: Label in the edges connecting to check node $\mathtt{c}$ and variable node $\mathtt{v}$
$\mathtt{V}$: Set of the variable nodes
$\mathcal{N}(\mathtt{c})$: Set of nodes connecting to the check node $\mathtt{c}$

$$HE := \{Hv \mid v\in E\}, \quad \text{for invertible matrix } H \text{ and } E\in\mathbb{F}_2^m,$$

$$\sum_{i=1}^m E_i := \{\sum_j v_j \mid v_j \in E_j\}, \quad \forall E_i \in \mathbb{F}_2^m.$$

Stopping Constellation Distribution

Weight of stopping constellation

$$\ell := \{\mathtt{v} \in \mathtt{V} \mid \dim E_{\mathtt{v}} \neq 0\}$$

$A^{\mathtt{G}}(\ell)$: The number of stopping constellation of weight ℓ for $\mathtt{G} \in \mathrm{EGL}(N, \lambda, \rho, m)$

Average stopping constellation distribution

$$A(\ell) := \sum_{\mathtt{G} \in \mathrm{EGL}(N,\lambda,\rho,m)} \frac{A^{\mathtt{G}}(\ell)}{|\mathrm{EGL}(N, \lambda, \rho, m)|}$$

Average Stopping Constellation Distribution

[Theorem 1] Average Stopping Constellation Distribution

$$A(\ell) = \sum_{\boldsymbol{b}:\sum_i b_i=\xi} \frac{\mathrm{coef}((Q(\boldsymbol{s},t)P(\boldsymbol{u}))^N, t^\ell \prod_{i=1}^m s_i^{b_i} u_i^{b_i})}{\binom{\xi}{b_0,b_1,\ldots,b_m} \prod_{k=1}^m \begin{bmatrix} m \\ k \end{bmatrix}^{b_k}},$$

$$Q(\boldsymbol{s},t) := \prod_j \left\{1 + t\sum_{i=1}^m \begin{bmatrix} m \\ i \end{bmatrix} s_i^j \right\}^{L_j}, \quad P(\boldsymbol{u}) := \prod_k \{f_k(\boldsymbol{u})\}^{R_k(1-r)},$$

$\mathrm{coef}(g(\boldsymbol{s},t,\boldsymbol{u}), t\prod_{i=1}^m s_i^{b_i} u_i^{b_i})$: coefficient of $t\prod_{i=1}^m s_i^{b_i} u_i^{b_i}$ for polynomial $g(\boldsymbol{s},t,\boldsymbol{u})$.

$$[m] := \prod_{i=1}^m (2^i - 1), \quad \begin{bmatrix} m \\ i \end{bmatrix} := \frac{[m]}{[i][m-i]}, \quad \binom{\xi}{b_0,b_1,\ldots,b_m} := \frac{\xi!}{b_0! b_1! \cdots b_m!},$$

$$f_k(\boldsymbol{u}) = \sum_{\boldsymbol{d}:\sum_i d_i = k} \binom{k}{d_0,d_1,\ldots,d_m} \prod_{i=1}^m u_i^{d_i} \sum_{\boldsymbol{a}_k \in D_k} \frac{[m]}{\prod_{S\subset[1,k]}[a_k(S)]} 2^{T_k},$$

$$T_k := \frac{1}{2} \sum_{S_1,S_2\subset[1,k]:S_1\not\subset S_2, S_1\not\supset S_2} a_k(S_1)a_k(S_2).$$

Let p_j be the smallest integer s.t. $j \le \sum_{i=0}^{p_j} d_i$,

$D_k := \{\boldsymbol{a}_k \mid \sum_{S\subset[1,k]} a_k(S) = m, a_k(S) = 0 \text{ for } |S| = k-1, \sum_{S:j\in S} a_k(S) = m - p_j\}$.

Sketch of Proof for Theorem 1

ℓ : weight of stopping constellation

b_i : the number of edges connecting to variable nodes with dimension i

1. For a given $\ell, \boldsymbol{b}$, the number of constellations of linear subspaces satisfying constraints for variable nodes is
   $\mathrm{coef}(Q(\boldsymbol{s}, t)^N, t^\ell \prod_{i=1}^{m} s_i^{b_i})$
2. For a given $\boldsymbol{b}$, the number of constellations of linear subspaces satisfying $E_{\mathtt{v}} \subseteq H_{\mathtt{c},\mathtt{v}}^{-1}\left(\sum_{i \in \mathcal{N}(\mathtt{c}) \setminus \{\mathtt{v}\}} H_{\mathtt{c},i} E_i\right)$ for all the check nodes
   $\mathrm{coef}(P(\boldsymbol{u})^N, u_i^{b_i})$
3. For a given $\boldsymbol{b}$, the number of edges satisfying constraints is
   $\prod_{i=0}^{m} b_i!([m-i][i])^{b_i}$
4. The number of elements in $\mathrm{EGL}(N, \lambda, \rho, m)$ is
   $\xi![m]^\xi$

Asymptotic Analysis 1 : Growth Rate

Define the normalized weight as $\omega := \ell/N$, $\beta_i := b_i/N$.

[Definition] Growth rate

$$\gamma(\omega) := \lim_{N\to\infty} \frac{1}{N} \log A(\omega N)$$

[Theorem 2] Growth rate of average number of stopping constellation

$$\begin{aligned}\gamma(\omega) &= \sup_{\boldsymbol{\beta}>\mathbf{0}} \inf_{\boldsymbol{s}>\mathbf{0},t>0,\boldsymbol{u}>\mathbf{0}} \Big\{ \log Q(\boldsymbol{s},t) + \log P(\boldsymbol{u}) - \omega \log t \\ &\qquad - \textstyle\sum_{i=1}^{m} \beta_i \log \begin{bmatrix} m \\ i \end{bmatrix} s_i u_i + \sum_{i=0}^{m} \beta_i \log \beta_i - \log \epsilon \Big\} \\ &= \sup_{\boldsymbol{\beta}>\mathbf{0}} \inf_{\boldsymbol{s}>\mathbf{0},t>0,\boldsymbol{u}>\mathbf{0}} \hat{\gamma}(\omega,\boldsymbol{s},t,\boldsymbol{u}).\end{aligned}$$

A point which achieves the minimum of $\hat{\gamma}(\omega,\boldsymbol{s},t,\boldsymbol{u})$ is the solution of

$$\beta_i = \frac{s_i}{Q}\frac{\partial Q}{\partial s_i}, \qquad \omega = \frac{t}{Q}\frac{\partial Q}{\partial t}, \qquad \beta_i = \frac{u_i}{P}\frac{\partial P}{\partial u_i}$$

Sketch of proof for Theorem 2

[Burshtein 2004] Theorem 3

$$\lim_{N\to\infty} \frac{1}{N} \log \mathrm{coef}(p(x_1, \ldots, x_m), (x_1^{\alpha_1} \cdots x_m^{\alpha_m})^N)$$
$$= \inf_{x_1,\ldots,x_m} \log \frac{p(x_1, \ldots, x_m)}{x_1^{\alpha_1} \cdots x_m^{\alpha_m}}$$

A point $\boldsymbol{x}$ which achieves the minimum of $\log \frac{p(x_1,\ldots,x_m)}{x_1^{\alpha_1}\cdots x_m^{\alpha_m}}$ is the solution of $\forall k \in \{1, 2, \ldots, m\}$

$$\frac{x_k}{p(x_1, \ldots, x_m)} \frac{\partial p(x_1, \ldots, x_m)}{\partial x_k} = \alpha_k$$

From this theorem and Theorem 1, we have Theorem 2.

[Burshtein 2004] D. Burshtein, and G. Miller, "Asymptotic enumeration method for analyzing LDPC codes," IEEE Trans. on Inform. Theory, Jun. 2004

Asymptotic Analysis 2 : Differential of growth rate

[Lemma 1] Differential of growth rate

$$\frac{\mathrm{d}\gamma(\omega)}{\mathrm{d}\omega} = -\log t(\omega).$$

[Note]: the paramters $t, \boldsymbol{\beta}, \boldsymbol{u}, \boldsymbol{s}$ depend only on ω

[Theorem 3] Small weight Stopping Constellations

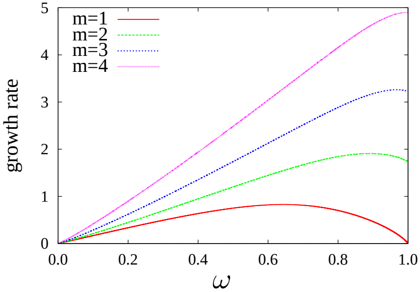
For $N \to \infty$, $\omega \to 0$ and $\mathrm{EGL}(N, \lambda, \rho, m)$ s.t. $\lambda_2 > 0$,

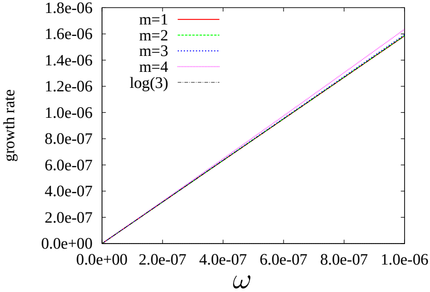
$$\gamma(\omega) = \log[\lambda'(0)\rho'(1)]\omega + o(\omega).$$

[Discussion] For small ω

- If $\lambda'(0)\rho'(1) > 1$, then $\gamma(\omega) > 0$
- If $\lambda'(0)\rho'(1) < 1$, then $\gamma(\omega) < 0$

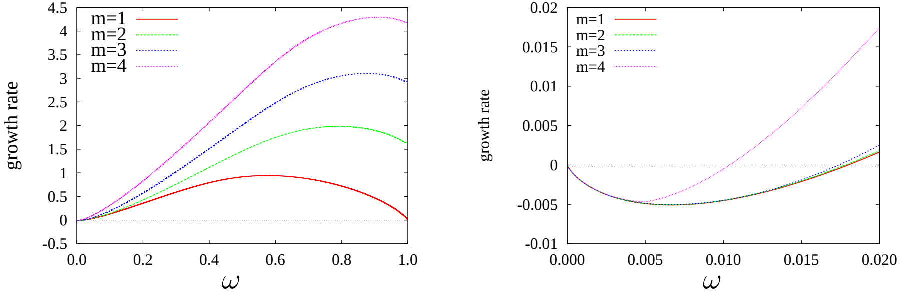
Numerical example (1)





Growth rate for the (2,4)-regular non-binary LDPC ensembles

Numerical example (2)



Growth rate for the (3,6)-regular non-binary LDPC ensembles

Conclusion and future work

Conclusion

For irregular non-binary LDPC codes defined over general linear group,

- we give the stopping constellation distributions
- we derive the growth rate for the average of stopping constellation distributions

Future work

- To derive the stopping constellation distribution for irregular non-binary LDPC codes defined over finite field.