Analysis of Error Floors of Generalized Non-binary LDPC Codes over q-ary Memoryless Symmetric Channels

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Outline

Purposes of Research

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For non-binary low-density parity-check (LDPC) matrices over general linear (GL) groups over the q-ary memoryless symmetric (q-MS) channels under belief propagation (BP) decoding,

- we propose a method to lower the error floors,
- we analyze the decoding error rates in the error floors.

Contribution of the research

	MBIOS channel	q-MS channel
LDPC matrices over GF	[NKS2011]	This research
LDPC matrices over GL group	This research	This research

GF : Galois field

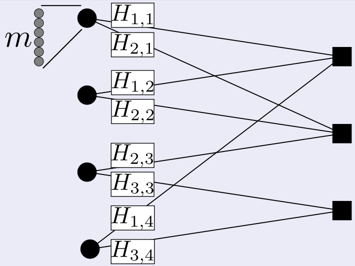
MBIOS channel : Memoryless binary-input output-symmetric channel

[NKS2011] T. Nozaki, K. Kasai, and K. Sakaniwa, "Analysis of Error Floors of Non-binary LDPC Codes over MBIOS Channel," *IEICE Trans. Fundamentals,* vol. E94-A, no. 11, pp.2144–2152, Nov. 2011.

Non-binary LDPC Codes

GL group $\mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$ is the set of $m_3 \times m_3$ invertible matrices over $\mathbb{F}_{2^{m_4}}$.

Non-binary LDPC matrices over general linear group



$$\left\{(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) \in (\mathbb{F}_2^m)^N \mid \sum_{i \in \mathcal{N}_c(j)} H_{j,i} \boldsymbol{x}_i^T = \mathbf{0}^T \ \forall j \in \{1, \ldots, M\}\right\}$$

Let $\mathcal{N}_c(j)$ be the set of variable nodes connecting to the j-th check node.

	$H_{i,j} \in \mathbb{F}_{2^m}$	$H_{i,j} \in \mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$	$H_{i,j} \in \mathrm{GL}(m, \mathbb{F}_2)$
	matrices over GF	matrices over GL group	
Decoding complexity	low	middle	high
Performance on waterfall	low	unknown	high
Performance on error floor	unknown	unknown	unknown

Channel Model

q-MS channel [HSS2008]

$\mathcal{X}$: input alphabet $\qquad$ $\mathcal{Y}$: output alphabet

q-ary memoryless channel is *symmetric* if there exists $\mathcal{T} : \mathcal{Y} \times \mathcal{X} \to \mathcal{Y}$ s.t.

- $\forall x \in \mathcal{X}$, $\mathcal{T}(\cdot, x) : \mathcal{Y} \to \mathcal{Y}$ is bijection
- $\forall x \in \mathcal{X}$, Jacobian of $\mathcal{T}(\cdot, x) : \mathcal{Y} \to \mathcal{Y}$ is 1 (if $\mathcal{Y}$ is continuous)
- $\forall x_1, x_2 \in \mathcal{X}$, $\forall y \in \mathcal{Y}$, $\quad p(y \mid x_1) = p(\mathcal{T}(y, x_2 - x_1) \mid x_2)$

[Lemma 1] All-zero codeword assumption

The decoding error rate of the LDPC code over $\mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$ through the 2^{m_1}-MS channel under BP decoding is independent of the sending codeword.

[HSS2008] E. Hof, I. Sason, and S. Shamai, "Performance bounds for non-binary linear block codes over memoryless symmetric channels," IEEE trans. on IT, Mar. 2008

Examples of q-MS channels

Examples of q-MS channels

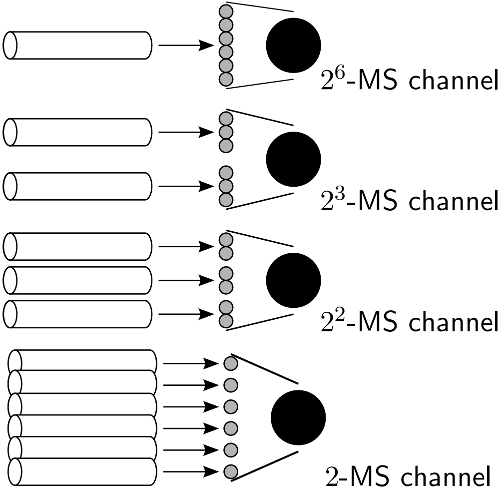
- memoryless binary input output symmetric (MBIOS) channels
  - binary erasure channel (BEC)
  - binary symmetric channel (BSC)
  - binary additive white Gaussian noise (BAWGN) channel
- q-ary symmetric channel (q-SC)

$$p(y \mid x) = \begin{cases} 1-\epsilon, & \text{if } \; y = x, \\ \epsilon/(q-1), & \text{if } \; y \neq x. \end{cases}$$

Channel Outputs to a Variable Node

We parameterize the number of channel outputs assigned to a variable node.

In the case for $m = 6$



Outline

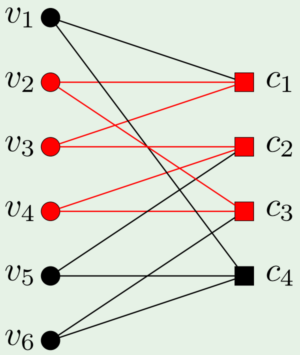
Error Floors and Zigzag Cycles

Error floors are mainly caused by small weight errors.

[Definition] Zigzag cycle

Zigzag cycles are circuit in the Tanner graph such that all the variable nodes are of degree two.

Property: Zigzag cycles cause small weight errors.



Methods to Lower Error Floors

To lower the error floors, we need to optimize the zigzag cycle.

Optimization of zigzag cycles

- Optimize the structure of Tanner graph
  $\rightarrow$ (e.g.) Progressive edge growth [HEA2005] removes the zigzag cycles of small weight from Tanner graph.
- Optimize the labels in the zigzag cycles
  $\rightarrow$ This research optimizes the labels in the zigzag cycles to lower the decoding error rates in the zigzag cycles.

[HEA2005] X.Y. Hu, E. Eleftherious, and D. Arnold, "Regular and irregular progressive edge-growth Tanner graphs," *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 386–396, Jan. 2005

Optimization of Labels to Lower Error Floors

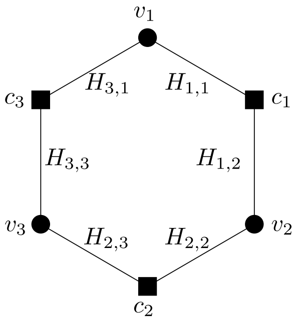
For zigzag cycle with labels $H_{1,1}, H_{1,2}, H_{2,2}, \ldots, H_{w,w}, H_{w,1}$,
define $\chi := H_{1,1}^{-1} H_{1,2} H_{2,2}^{-1} \cdots H_{w,w}^{-1} H_{w,1}$.
The *order* σ of χ is the smallest integer s.t. χ^{σ} is identify matrix.

Label selection to lower the error floors

Optimize labels in the zigzag cycles satisfying $\sigma = 2^{m_3 m_4} - 1$ to lower the error floors for non-binary LDPC code over $\mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$.

Outline of proof

1. [Theorem 1] shows the condition for the successful decoding of the zigzag cycles.
2. [Corollary 1] gives the zigzag cycles which have the best decoding performance.
3. [Lemma 2] gives the relation between condition in [Corollary 1] and order of χ.



Outline of Proof (1)

[Definition] Orbit

Let $\langle \chi \rangle$ be the cyclic subgroup generated by $\chi \in \mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$, i.e., $\langle \chi \rangle := \{\chi^j \mid j = 0, 1, \ldots\}$.
Define the *orbit* of $x \in \mathbb{F}_{2^{m_4}}^{m_3}$ under $\langle \chi \rangle$ as $\langle \chi \rangle x := \{gx \mid g \in \langle \chi \rangle\}$.

The set of orbits of $x \in \mathbb{F}_{2^{m_4}}^{m_3} \setminus \{0\}$ under $\langle \chi \rangle$ forms *partition* of $\mathbb{F}_{2^{m_4}}^{m_3} \setminus \{0\}$.
A *set of class representatives* S_χ is a subset of $\mathbb{F}_{2^{m_4}}^{m_3} \setminus \{0\}$ which contains exactly one elements from each orbit.

Outline of Proof (2)

Let $C_i(x)$ be the initial message from i-th variable node in the BP decoder for $x \in \mathbb{F}_{2^{m_4}}^{m_3}$.

[Theorem 1] Condition for successful decoding

Consider the zigzag cycle of weight w with $\chi = H_{1,1}^{-1} H_{1,2} \cdots H_{w,w}^{-1} H_{w,1}$.
All the symbols in the zigzag cycles are successful decoded *iff*

$$\prod_{t=0}^{|\langle\chi\rangle x|-1} \prod_{s=1}^{w} C_s(0) > \prod_{t=0}^{|\langle\chi\rangle x|-1} \prod_{s=1}^{w} C_s\left(\left(\prod_{j=s}^{w} \iota_j\right) \chi^t x\right) \quad \forall x \in S_\chi.$$

No symbols in the zigzag cycles are successful decoded *iff*

$$\prod_{t=0}^{|\langle\chi\rangle x|-1} \prod_{s=1}^{w} C_s(0) \leq \prod_{t=0}^{|\langle\chi\rangle x|-1} \prod_{s=1}^{w} C_s\left(\left(\prod_{j=s}^{w} \iota_j\right) \chi^t x\right) \quad \exists x \in S_\chi.$$

Outline of Proof (3)

[Corollary 1] Zigzag cycles having best performance

The zigzag cycles with χ s.t. $|S_\chi| = 1$ have the best decoding performance in the zigzag cycles for a fixed weight.

[Lemma 2] Relation between orbit and order

The order of $\chi \in \mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$ is $2^{m_3 m_4} - 1$ *iff* $|S_\chi| = 1$.

The zigzag cycles with χ of order $2^{m_3 m_4} - 1$ have the best decoding performance in the zigzag cycles for a fixed weight.

Outline

Condition for Decoding Failure

[Definition] Log-likelihood ratio (LLR)

Assume the 2^{m_1}-MS channel.
For $\gamma \in \mathcal{X} = \mathbb{F}_2^{m_1}$, the LLR $Z_{v,i}(y_{v,i}, \gamma)$ corresponding to the i-th channel output $y_{v,i}$ in the v-th variable node is defined as

$$Z_{v,i}(y_{v,i}, \gamma) = \log \frac{p(y_{v,i} \mid 0)}{p(y_{v,i} \mid \gamma)}.$$

[Corollary 2] Condition for decoding failure for zigzag cycles

We consider the zigzag cycle of weight w with the matrix χ of order $2^{m_3 m_4} - 1$.
No symbols in the zigzag cycle are successfully decoded *iff*

$$\sum_{v=1}^{w} \sum_{i=1}^{m_2} \sum_{\gamma \in \mathbb{F}_2^{m_1} \setminus \{0\}} Z_{v,i}(y_{v,i}, \gamma) \leq 0.$$

L-Density and Bhattacharyya Functional

[Definition] L-density for 2^{m_1}-MS channels

Define the random variable

$$L(Y) := \sum_{\gamma \in \mathbb{F}_2^{m_1} \setminus \{0\}} \log \frac{p(Y \mid 0)}{p(Y \mid \gamma)}.$$

Let $\mathsf{a}(x)$ be the density function of $L(Y)$. We refer $\mathsf{a}(x)$ as *L-density*.

The 2^{m_1}-MS channels are characterized by L-density.

[Definition] Bhattacharyya functional

For a L-density $\mathsf{a}(x)$, the Bhattacharyya functional $\mathfrak{B}(\mathsf{a})$ is defined as

$$\mathfrak{B}(\mathsf{a}) = \int_{-\infty}^{\infty} \mathsf{a}(x) \exp[-x/2] dx.$$

Decoding Error Rate of Zigzag Cycle

[Corollary 3] Decoding error rate of zigzag cycle

Let $m = m_1 m_2 = m_3 m_4$.
Let $\mathrm{P}_{\mathrm{zc}}(w, m_1, m_2, \mathsf{a})$ be the symbol error rate for the zigzag cycle of weight w with χ of order $2^m - 1$ over the 2^{m_1}-MS channel with L-density $\mathsf{a}(x)$ under BP decoding.
Define $Z^{(k)} = \sum_{i=1}^{k} Z_i$, where $Z_1, Z_2, \ldots, Z_k$ denote i.i.d. random variable with L-density $\mathsf{a}(x)$. Then

$$\mathrm{P}_{\mathrm{zc}}(w, m_1, m_2, \mathsf{a}) = \Pr(Z^{(wm_2)} \leq 0) \leq \mathfrak{B}(\mathsf{a})^{wm_2}.$$

Note: For a fixed $m = m_3 m_4$, the decoding error rate of zigzag cycles are independent from m_3 and m_4.

Lower Bound of Error Floor

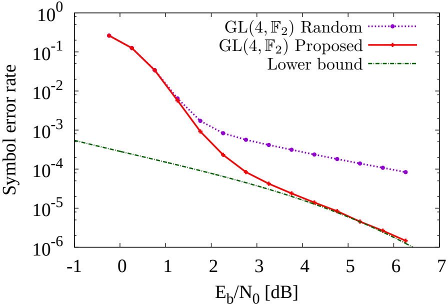
Consider the LDPC codes designed by proposed method.

[Theorem 2] Lower bound of error floors for non-binary LDPC codes

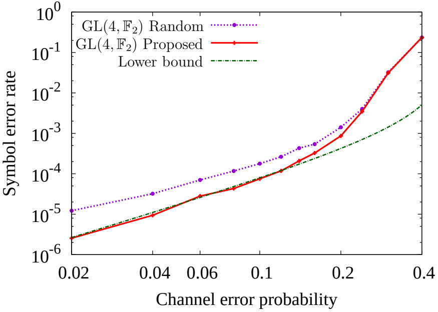
Let w_{g} be the minimum weight of the zigzag cycle in the LDPC code ensemble.
Define $\mu := \lambda'(0)\rho'(1)$, where (λ, ρ) is a pair of degree distribution.
For sufficiently large N and $\mathfrak{B}(\mathsf{a}) < \mu^{-1/m_2}$, the symbol error rate of the LDPC code ensemble is lower bounded by

$$\frac{1}{2N} \sum_{w=w_{\mathrm{g}}} \mu^{w} \Pr(Z^{(wm_2)} \leq 0).$$

Simulation Result (1)



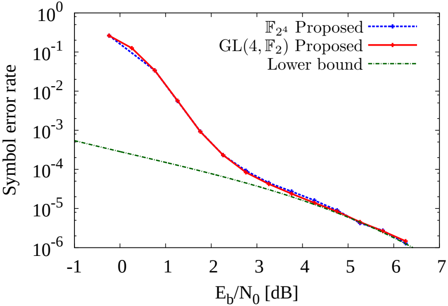
BAWGN channel case ($w_{\rm g} = 1$)



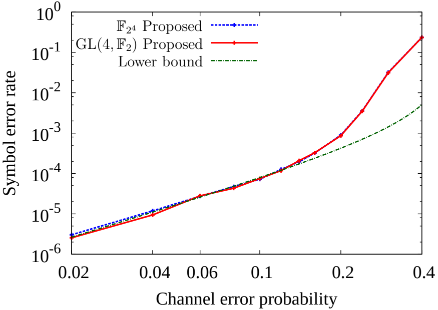
2^4-SC case ($w_{\rm g} = 2$)

(2,3)-regular LDPC code over $\mathrm{GL}(4, \mathbb{F}_2)$ code length 1260 bit.

Simulation Result (2)



BAWGN channel case ($w_{\mathrm{g}} = 1$)



2^4-SC case ($w_{\mathrm{g}} = 2$)

(2,3)-regular LDPC code over $\mathbb{F}_{2^4}$ and $\mathrm{GL}(4, \mathbb{F}_2)$ code length 1260 bit.

Conclusion

Conclusion

For non-binary LDPC codes over $\mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$ over the 2^{m_1}-MS channels under BP decoding,

- we proposed a method to lower the error floors,
- we analyzed the decoding error rates in the error floors.

	$H_{i,j} \in \mathbb{F}_{2^m}$	$H_{i,j} \in \mathrm{GL}(m_3, \mathbb{F}_{2^{m_4}})$	$H_{i,j} \in \mathrm{GL}(m, \mathbb{F}_2)$
	codes over GF	codes over GL group	
Decoding complexity	low	middle	high
Performance on waterfall	low	unknown	high
Performance on error floor	same		

Future works

- Improve the decoding error rates in the waterfall regions.