Message Passing Algorithm with MAP Decoding on Zigzag Cycles for Non-binary LDPC Codes

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Introduction

We propose a new decoding algorithm for non-binary low-density parity-check (LDPC) codes over binary erasure channel (BEC).

Strength of the proposed algorithm

- In the error floor region, decoding erasure rate of proposed algorithm is lower than that of BP decoder
- Message passing algorithm

Weakness of the proposed algorithm

- Decoding complexity is higher than the BP decoder

Main idea

- Reduce the decoding erasures in the zigzag cycles
- Three phase decoding algorithm
  - [Phase 1] BP decoding
  - [Phase 2] Detection of the zigzag cycles with erasures
  - [Phase 3] MAP decoding on the zigzag cycles

Outline

1. Introduction
2. Non-binary LDPC Codes and Zigzag Cycles
3. Message Passing Algorithm with MAP Decoding on Zigzag Cycles
   - Zigzag Cycle Detection
   - MAP Decoding on Zigzag Cycles
   - Simulation Result
4. Conclusion

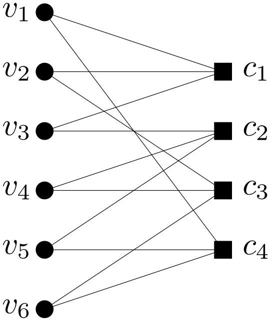
Non-binary LDPC Code

Non-binary LDPC Code

A linear code defined by sparse parity check matrix $H \in \mathbb{F}_{2^m}^{M \times N}$

$$C := \left\{ \boldsymbol{x} \in \mathbb{F}_{2^m}^{N} \mid H\boldsymbol{x}^T = \mathbf{0}^T \right\}$$

It is known that the optimized irregular non-binary LDPC codes contain variable nodes of degree 2.



$$\begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} & \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & 0 & 0 & 0 \\ 0 & 0 & h_{2,3} & h_{2,4} & h_{2,5} & 0 \\ 0 & h_{3,2} & 0 & h_{3,4} & 0 & h_{3,6} \\ h_{4,1} & 0 & 0 & 0 & h_{4,5} & h_{4,6} \end{pmatrix} \end{array}$$

Error Floors of Non-binary LDPC Codes

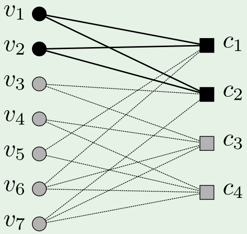
Decoding erasures with small weight cause error floors.

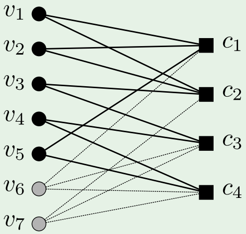
[Definition] Zigzag Cycle

A zigzag cycle $(\mathsf{V}_{\mathrm{zc}} \cup \mathsf{C}_{\mathrm{zc}}, \mathsf{E}_{\mathrm{zc}})$ is a simple cycle such that the degrees of variable nodes in V_{zc} are two in the Tanner graph. All the check nodes in C_{zc} connect to the variable nodes V_{zc} exactly twice.

- The set of the variable nodes in V_{zc} forms stopping sets.
- Zigzag cycles cause decoding erasures with small weight.

[Example] Zigzag Cycle





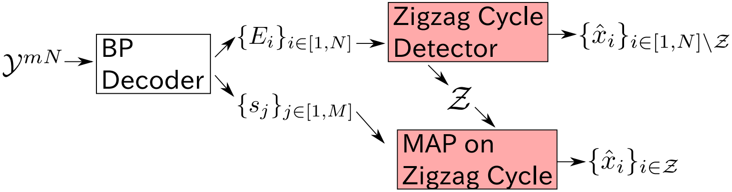
(Left) A zigzag cycle
(Right) A connected graph which is not zigzag cycle

Outline

Abstract of Proposed Decoding Algorithm

Key Concept of Decoding Algorithm

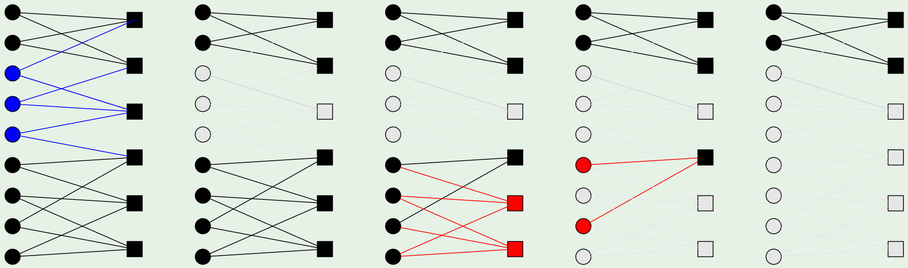
The BP decoder can not recover zigzag cycles in which all the bits are erased, but MAP decoder can recover those.
Combine "BP decoder" with "MAP decoding on the zigzag cycles"



- $\mathcal{Y}^{mN} \in \{0,1,?\}^{mN}$: channel outputs.
- $E_i \subseteq \mathbb{F}_{2^m}$: BP decoding result for the i-th variable node (VN).
- $s_j := \sum_{i\in\{k||E_k|=1\}} h_{j,i}\hat{x}_i^{\mathrm{BP}} \in \mathbb{F}_{2^m}$: syndrome of the j-th check node (CN).
- $\mathcal{Z} \in [1,N]$: the set of VNs in zigzag cycles with BP decoding erasures.
- $\hat{x}_i \in \mathbb{F}_{2^m} \cup \{?\}$: the decoding result of the i-th VN.

Zigzag Cycle Detection (1:Example)

If the i-th BP decoding result E_i is not unique, then the i-th VN has BP decoding erasure.



(1)

(1) The blue VNs are correctly decoded by BP decoder

Zigzag Cycle Detection (2:Algorithm)

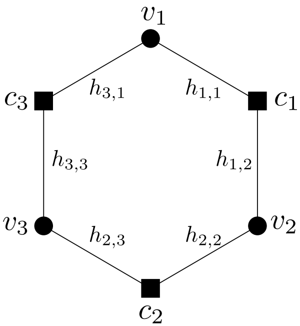
[Algorithm 1] Zigzag Cycle Detection

1. (Construction of residual graph)
   Set G as the Tanner graph. For $i \in [1, N]$, if $|E_i| = 1$, i-th VN and adjacent edges are removed from G and $\hat{x}_i = \gamma$ where γ is the unique element of E_i.
2. (Removing CNs of residual degrees more than 2)
   For $j \in [1, M]$, if the j-th CN is of degree more than 2 in G, $\hat{x}_i =?$ and remove the VNs connecting to the j-th CN and edges adjacent to those VNs from G.
3. (Removing CNs whose residual degrees decrease)
   For $j \in [1, M]$, if the residual degrees of the j-th CN decreases, $\hat{x}_i =?$ and remove the VNs connecting to the j-th CN and edges adjacent to those VNs from G.
4. (Decision)
   If the residual degrees of all the CNs do not decrease in Step 3, let $\mathcal{Z}$ be the set of variable nodes in G and algorithm stops. Otherwise, go to Step 3.

MAP Decoding on Zigzag Cycles (1:Inverse Matrix)

To simplify the notations, we consider the zigzag cycle of weight 3.

Let s_i be syndrome of the i-th check node.
$\zeta_i := h_{i-1,i} h_{i,i}^{-1}$, $\beta := \prod_{i=1}^{3} \zeta_i$.
Assume that the submatrix corresponding to zigzag cycle is non-singular, i.e., $\beta \neq 1$.



$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} h_{1,1} & h_{1,2} & 0 \\ 0 & h_{2,2} & h_{2,3} \\ h_{3,1} & 0 & h_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\iff \frac{1}{1+\beta} \begin{pmatrix} h_{1,1}^{-1} & h_{1,1}^{-1}\zeta_2 & h_{1,1}^{-1}\zeta_2\zeta_3 \\ h_{2,2}^{-1}\zeta_3\zeta_1 & h_{2,2}^{-1} & h_{2,2}^{-1}\zeta_3 \\ h_{3,3}^{-1}\zeta_1 & h_{3,3}^{-1}\zeta_1\zeta_2 & h_{3,3}^{-1} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

MAP Decoding on Zigzag Cycles (2:Decoding Result)

Hence, the MAP decoding results are

$$\hat{x}_1 = (1+\beta)^{-1} h_{1,1}^{-1}(s_1 + \zeta_2 s_2 + \zeta_2\zeta_3 s_3) =: (1+\beta)^{-1} A_1,$$
$$\hat{x}_2 = (1+\beta)^{-1} h_{2,2}^{-1}(\zeta_3\zeta_1 s_1 + s_2 + \zeta_3 s_3) =: (1+\beta)^{-1} A_2,$$
$$\hat{x}_3 = (1+\beta)^{-1} h_{3,3}^{-1}(\zeta_1 s_1 + \zeta_1\zeta_2 s_2 + s_3) =: (1+\beta)^{-1} A_3.$$

Define $B_i := \beta^{-1} A_i$. $\beta = A_i/B_i$ $\beta \neq 1 \iff A_i \neq B_i$.
Then, we get

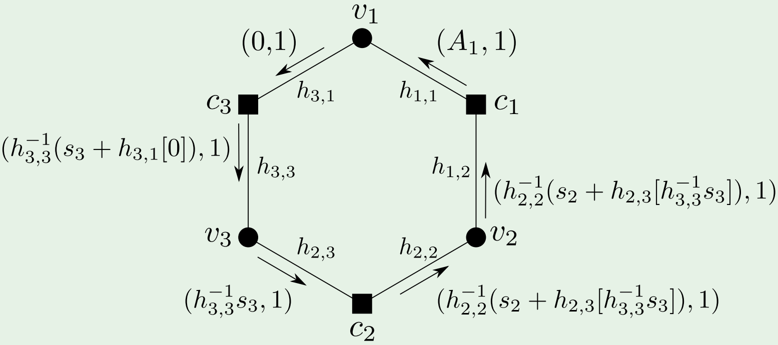
$$\hat{x}_i = \frac{A_i}{1+\beta} = \frac{A_i}{1 + A_i/B_i} = \frac{A_i B_i}{A_i + B_i}.$$

Hence, we obtain the MAP decoding result if we get A_i and B_i.

A message passing algorithm is able to calculate A_i and B_i.

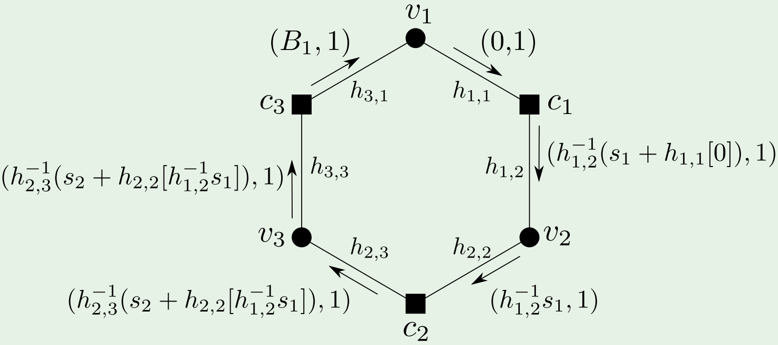
MAP Decoding on Zigzag Cycle (3:Example)

$$
\begin{aligned}
A_1 &= h_{1,1}^{-1}\Big(s_1 + \zeta_2\big(s_2 + \zeta_3 s_3\big)\Big) \\
&= h_{1,1}^{-1}\Big(s_1 + h_{1,2}\underline{h_{2,2}^{-1}\big(s_2 + h_{2,3}\underline{h_{3,3}^{-1}(s_3 + h_{3,1}\underline{0}_0)}_1\big)}_2\Big).
\end{aligned}
$$

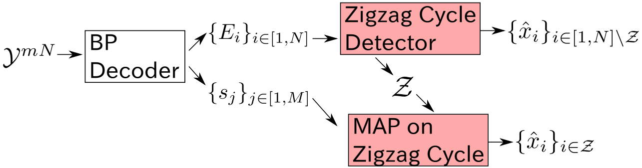


MAP Decoding on Zigzag Cycle (4:Example)

$$
\begin{aligned}
B_1 &= h_{3,1}^{-1}\Big(s_3 + \zeta_3^{-1}\big(s_2 + \zeta_2^{-1} s_1\big)\Big) \\
&= h_{3,1}^{-1}\Big(s_3 + h_{3,3}{\underline{h_{2,3}^{-1}\big(s_2 + h_{2,2}{\underline{h_{1,2}^{-1}(s_1 + h_{1,1}{\underline{0}}_0)}}_1\big)}}_2\Big).
\end{aligned}
$$



MAP Decoding on Zigzag Cycle (5:Algorithm)



- $s_j \in \mathbb{F}_{2^m}$: syndrome of the j-th check node (CN).
- $\mathcal{Z} \in [1, N]$: the set of VNs in zigzag cycles with BP decoding erasures.
- $\hat{x}_i \in \mathbb{F}_{2^m} \cup \{?\}$: the decoding result of the i-th VN.

- $(\psi^{(\ell)}_{i\to j}, p^{(\ell)}_{i\to j}) \in \mathbb{F}_{2^m} \times [1, w]$: the message from the i-th VN to the j-th CN at the ℓ-th iteration.
- $(\phi^{(\ell)}_{j\to i}, q^{(\ell)}_{j\to i}) \in \mathbb{F}_{2^m} \times [1, w]$: the message from the j-th CN to the i-th VN at the ℓ-th iteration.
- $\mathcal{N}_{\mathrm{v}}(i)$: the set of indices of CNs which are neighbors of the i-th VN.
- $\mathcal{N}_{\mathrm{c}}(j)$: the set of indices of VNs which are neighbors of the j-th CN.
- $\mathcal{N}_{\mathrm{v}}(\mathcal{Z})$: the set of indices of CNs which are neighbors of the VNs in $\mathcal{Z}$.

MAP Decoding on Zigzag Cycle (6:Algorithm)

[Algorithm 2] MAP Decoding on Zigzag Cycle

1. Set $\ell = 0$. For $i \in \mathcal{Z}$ and $j \in \mathcal{N}_{\mathrm{v}}(i)$, the i-th VN sends
$$(\psi_{i\to j}^{(0)}, p_{i\to j}^{(0)}) = (0, i).$$
2. For $j \in \mathcal{N}_{\mathrm{v}}(\mathcal{Z})$ and $i \in \mathcal{N}_{\mathrm{c}}(j) \cap \mathcal{Z}$, the j-th CN sends
$$(\phi_{j\to i}^{(\ell+1)}, q_{j\to i}^{(\ell+1)}) = (h_{j,i}^{-1}(s_j + h_{j,i'}\psi_{i'\to j}^{(\ell)}), p_{i'\to j}^{(\ell)}),$$
where i' is the unique element of $(\mathcal{N}_{\mathrm{c}}(j) \cap \mathcal{Z}) \setminus \{i\}$.
3. For $i \in \mathcal{Z}$, if $i = q_{j\to i}^{(\ell+1)} = q_{j'\to i}^{(\ell+1)}$ where $\{j, j'\} = \mathcal{N}_{\mathrm{v}}(i)$, then $\mathcal{Z} \leftarrow \mathcal{Z} \setminus \{i\}$
$$\hat{x}_i = \begin{cases} \phi_{j\to i}^{(\ell+1)}\phi_{j'\to i}^{(\ell+1)}/(\phi_{j\to i}^{(\ell+1)} + \phi_{j'\to i}^{(\ell+1)}), & \text{if } \phi_{j\to i}^{(\ell+1)} \neq \phi_{j'\to i}^{(\ell+1)}, \\ ?, & \text{if } \phi_{j\to i}^{(\ell+1)} = \phi_{j'\to i}^{(\ell+1)}. \end{cases}$$
4. If $\mathcal{Z} = \emptyset$, the algorithm stops.
5. Set $\ell \leftarrow \ell + 1$. For $i \in \mathcal{Z}$ and $j \in \mathcal{N}_{\mathrm{v}}(i)$, the i-th VN sends
$$(\psi_{i\to j}^{(\ell)}, p_{i\to j}) = (\phi_{j'\to i}^{(\ell)}, q_{j'\to i}),$$
where j' is the unique element of $\mathcal{N}_{\mathrm{v}}(i) \setminus \{j\}$. Go to Step 2.

Decoding Complexity (Worst case)

$\mathcal{E}$: the number of variable node with decoding erasure

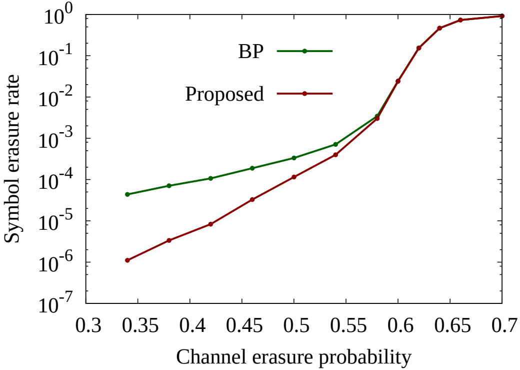
Zigzag cycle detection

(The number of iteration) $\leq \mathcal{E}/2$
(The number of edges conveying messages per iteration) $\leq 2\mathcal{E}$
$\Rightarrow$ Complexity $\mathcal{O}(\mathcal{E}^2)$

MAP decoding on zigzag cycle

(The number of iteration) $\leq \mathcal{E}$
(The number of calculations per iteration) $\leq \mathcal{O}(\mathcal{E})$
$\Rightarrow$ Complexity $\mathcal{O}(\mathcal{E}^2)$

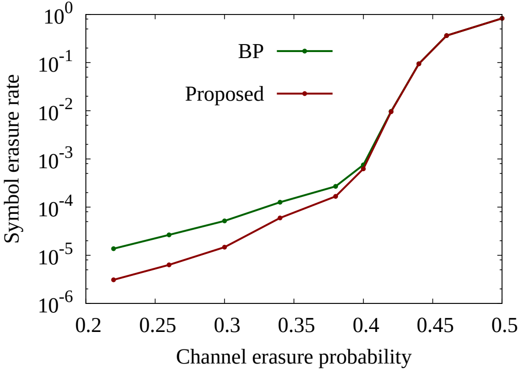
Simulation Result (1)



(2,3)-regular LDPC codes constructed by [NKS12]
$\mathbb{F}_{2^4}$
Symbol code length 315

[NKS12] T. Nozaki, K. Kasai, and K. Sakaniwa, "Analysis of Error Floors of Non-binary LDPC Codes over BEC," *IEICE Trans. Fundamentals.* vol. E95-A, no. 1, pp. 381–390, Jan. 2012

Simulation Result (2)



Irregular LDPC codes constructed [NKS12]

$\mathbb{F}_{2^4}$

Symbol code length 252

$\Lambda(x) =$
$195x^2 + 26x^3 + 29x^4 + 2x^5$
$P(x) = 36x^4 + 90x^5$

[NKS12] T. Nozaki, K. Kasai, and K. Sakaniwa, "Analysis of Error Floors of Non-binary LDPC Codes over BEC," *IEICE Trans. Fundamentals.* vol. E95-A, no. 1, pp. 381–390, Jan. 2012

Conclusion

Conclusions

- We have proposed a message passing decoding algorithm with MAP decoding on the zigzag cycles.

Remark

- The proposed algorithm is extended to the memoryless binary-input output-symmetric channels (IEICE transactions).