Analysis of Breakdown Probability of Wireless Sensor Networks with Unreliable Relay Nodes

Takayuki Nozaki[1], Takafumi Nakano[2], Tadashi Wadayama[2]

[1]Yamaguchi University

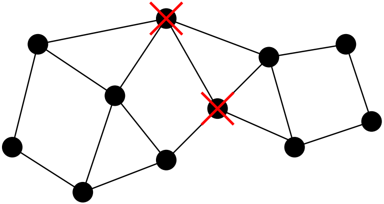
[2]Nagoya Institute of Technology

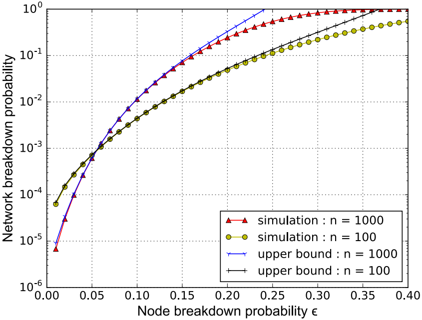
This work is supported by JSPS Grant-in-Aid for Scientific Research 16K16007 and 16H02878

ISIT2017

26th June 2017

Abstract





$$P_{\lambda,n}(\epsilon) \leq \sum_{j=0}^{n} \frac{\binom{n}{j}\epsilon^j(1-\epsilon)^{n-j}}{2\binom{\lambda n}{\lambda j}} \sum_{n_1=1}^{n-j-1} \frac{\binom{n-j}{n_1}}{\binom{\lambda(n-j)}{\lambda n_1}} \sum_{i_1=0}^{\lambda \min\{n_1,j\}} 2^{i_1}$$

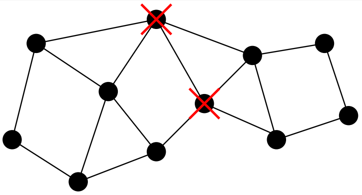
$$\times \binom{\frac{\lambda n}{2}}{i_1,\ \frac{\lambda n_1 - i_1}{2},\ \frac{\lambda(n-n_1)-i_1}{2}} \binom{\lambda(n-n_1)-i_1}{\lambda(n-n_1-j)}.$$

Background (1)

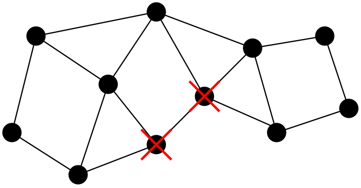
Wireless sensor networks (WSNs) are represented by graphs
There are possibilities that nodes are breakdown (or flat batteries)

The network is breakdown if survivor graph is separated
Survivor graph is a subgraph constructed from the remaining nodes

We analyze the network breakdown probability for WSNs with unreliable nodes



network is breakdown



network is not breakdown

Background (2)

Networks with unreliable *edges*

- Moore and Shannon [2] considered *network reliability problem*
- Provan and Ball [3] and Vailant [4] showed that network reliability problem is $\#\mathcal{P}$-complete
- Karger [5] presented a randomized polynomial time approximation algorithm for the all terminal network reliability problems

Networks with unreliable *nodes*

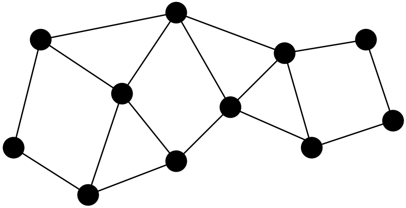
- There are no works, as far as the authors know.

Outline

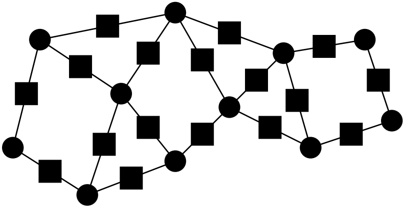
1. Models
   1. Random graph model
   2. Node fault model
2. Main result
   1. Main theorem (upper bound)
   2. Computer simulations
   3. Proof

Graph and Bipartite Graph

Any graph can be converted into a bipartite graph



Graph $\mathtt{G}$



Bipartite Graph $\mathtt{G}_\mathrm{b}$

Graph Ensemble and Node fault model

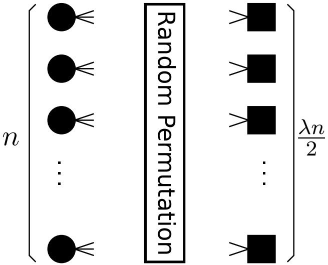
Define graph ensemble in a similar way to LDPC code ensemble

Graph Ensemble

$\mathcal{T}(n, \lambda, 2)$: $(\lambda, 2)$-regular bipartite graph ensemble
$\mathcal{G}(n, \lambda)$: counterpart of $\mathcal{T}(n, \lambda, 2)$ (λ-regular random graph ensemble)

Node fault model

Each node is independently broken w.p. ϵ.



$\mathcal{T}(n, \lambda, 2)$

Main Theorem

Theorem 1

$P_{\lambda,n}(\epsilon)$: ensemble average of network breakdown probability

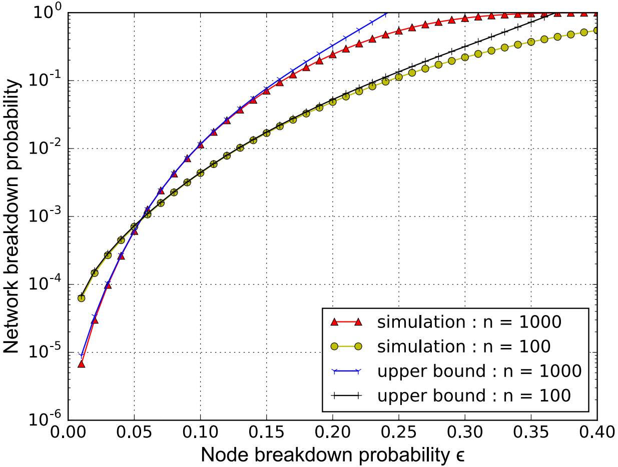
$$P_{\lambda,n}(\epsilon) \leq P^{(U)}_{\lambda,n}(\epsilon) := \sum_{j=0}^{n} \binom{n}{j} \epsilon^j (1-\epsilon)^{n-j} Q^{(U)}_{j,\lambda,n},$$

$$Q^{(U)}_{j,\lambda,n} := \frac{1}{2\binom{\lambda n}{\lambda j}} \sum_{n_1=1}^{n-j-1} \frac{\binom{n-j}{n_1}}{\binom{\lambda(n-j)}{\lambda n_1}} \sum_{i_1=0}^{\lambda \min\{n_1,j\}} 2^{i_1}$$
$$\times \binom{\frac{\lambda n}{2}}{i_1, \frac{\lambda n_1 - i_1}{2}, \frac{\lambda(n-n_1)-i_1}{2}} \binom{\lambda(n-n_1)-i_1}{\lambda(n-n_1-j)}.$$

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } k, n \in \mathbb{Z}^+, \\ 0, & \text{otherwise,} \end{cases}$$

$$\binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k} := \begin{cases} \frac{(a_1+a_2+\cdots+a_k)!}{a_1! a_2! \cdots a_k!}, & \text{if } a_1, a_2, \ldots, a_k \in \mathbb{Z}^+, \\ 0, & \text{otherwise.} \end{cases}$$

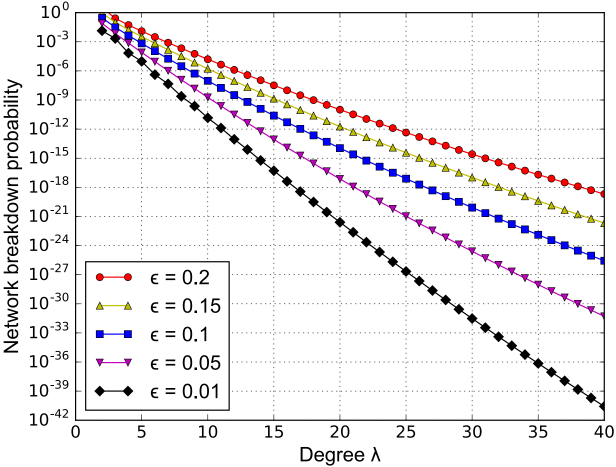
Computer experiments (1)



$$\lambda = 5,\ n = 100, 1000$$

- Upper bound is tight for small ϵ

Computer experiments (2)



$$n = 100$$

- Network breakdown probability appears an exponential function of λ

Proof (1: Ensemble average)

$\mathtt{G} \setminus \mathtt{Z}$: Survivor graph
$\mathbb{E}[\cdot]$: average over ensemble $\mathcal{G}(n, \lambda)$

$$
\begin{aligned}
P_{\lambda,n}(\epsilon) &= \sum_{j=0}^{n} \sum_{\mathtt{Z}\subset\mathtt{N},|\mathtt{Z}|=j} \mathsf{E}[\mathbb{I}[\mathtt{G} \setminus \mathtt{Z} : \text{separated}]]\epsilon^{j}(1-\epsilon)^{n-j} \\
&= \sum_{j=0}^{n} \sum_{\mathtt{Z}\subset\mathtt{N},|\mathtt{Z}|=j} Q_{j,\lambda,n}\epsilon^{j}(1-\epsilon)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \epsilon^{j}(1-\epsilon)^{n-j} Q_{j,\lambda,n}.
\end{aligned}
$$

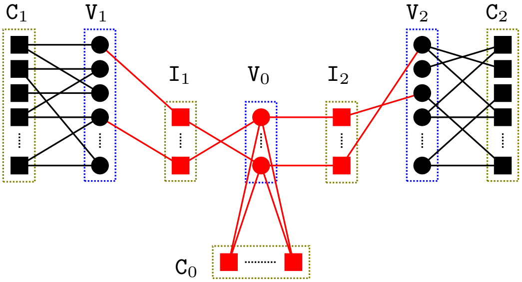
$Q_{j,\lambda,n}$: average of network breakdown probability when fixed j nodes are broken

$$
\begin{aligned}
Q_{j,\lambda,n} &:= \mathsf{E}[\mathbb{I}[\mathtt{G} \setminus \mathtt{Z} : \text{separated}]], \qquad (|\mathtt{Z}| = j) \\
&= \sum_{\mathtt{G}_{\mathrm{b}}\in\mathcal{T}(n,\lambda,2)} \frac{\mathbb{I}[(\mathtt{G} \setminus \mathtt{Z})_{\mathrm{b}} : \text{separated}]}{(n\lambda)!}
\end{aligned}
$$

Proof (2: Bipartite configuration)

It is not trivial to count $\sum_{\mathtt{G}_\mathrm{b} \in \mathcal{T}(n,\lambda,2)} \mathbb{I}[(\mathtt{G} \setminus \mathtt{Z})_\mathrm{b} : \text{separated}]$
We will derive upper bound by counting bipartite configurations

Bipartite configuration



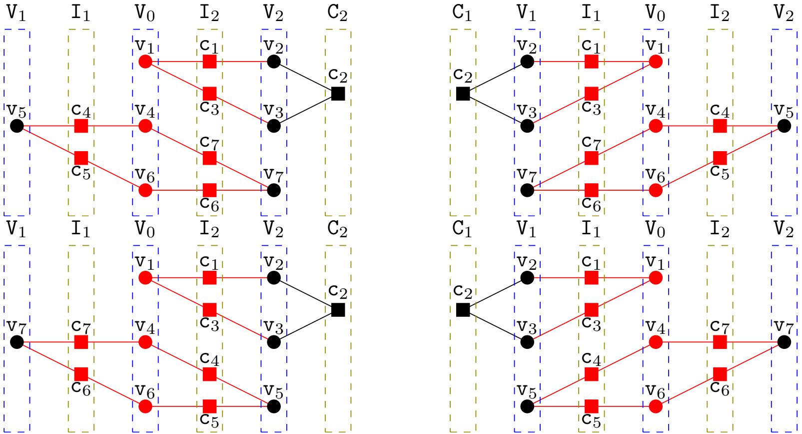
If nodes in $\mathtt{V}_0$ are broken, the network is breakdown
Bipartite configuration represents a bipartite graph that are separated if V_0 are removed

Proof (3: Bipartite configuration)

Each bipartite graph corresponds to at least 2 bipartite configurations

If count the bipartite configuration with certain condtion, we have an *upper bound* of $\sum_{\mathtt{G}_{\mathrm{b}} \in \mathcal{T}(n,\lambda,2)} \mathbb{I}[(\mathtt{G} \setminus \mathtt{Z})_{\mathrm{b}} : \text{separated}]$

Example: $n = 7$, $\lambda = 2$

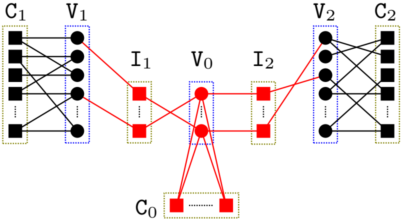


Proof (4: Number of bipartite configuration)

$n_0 := |V_0|, n_1 := |V_1|, i_1 := |I_1|, i_2 := |I_2|$
(Number of the other sets of nodes depends on n_0, n_1, i_1, i_2)

A_{n_0,n_1,i_1,i_2} : the number of bipartite configuration with n_0, n_1, i_1, i_2

$$A_{n_0,n_1,i_1,i_2} = (\lambda n)! \frac{\binom{n}{n_0,n_1,n_2}\binom{\lambda n/2}{i_1,i_2,c_0,c_1,c_2}}{\binom{\lambda n}{\lambda n_0,\lambda n_1,\lambda n_2}} 2^{i_1+i_2}.$$



Proof (5: Upper bound)

$K(j)$: the number of bipartite configurations $n_0 = j$

$$K(j) = \frac{1}{\binom{n}{j}} \sum_{n_1=1}^{n-j-1} \sum_{i_1=0}^{\lambda \min\{j,n_1\}} \sum_{i_2=0}^{\min\{\lambda n_2, \lambda j - i_1\}} A_{j,n_1,i_1,i_2}.$$

$$Q_{j,\lambda,n} = \frac{1}{(\lambda n)!} \sum_{\mathtt{G}_\mathrm{b} \in \mathcal{T}(n,\lambda,2)} \mathbb{I}[(\mathtt{G} \setminus \mathtt{Z})_\mathrm{b} : \text{separated}]$$
$$\leq \frac{1}{(\lambda n)!} \frac{K(j)}{2} =: Q^{(U)}_{j,\lambda,n}$$

(each bipartite graph corresponds to at least 2 bipartite configurations)

$$Q^{(U)}_{j,\lambda,n} := \frac{1}{2\binom{\lambda n}{\lambda j}} \sum_{n_1=1}^{n-j-1} \frac{\binom{n-j}{n_1}}{\binom{\lambda(n-j)}{\lambda n_1}} \sum_{i_1=0}^{\lambda \min\{n_1,j\}} 2^{i_1} \binom{\frac{\lambda n}{2}}{i_1, \frac{\lambda n_1 - i_1}{2}, \frac{\lambda(n-n_1)-i_1}{2}} \binom{\lambda(n-n_1)-i_1}{\lambda(n-n_1-j)}$$

Conclusion and on going work

Conclusion

- We have studied the network reliability problem under node fault model
- We have derived an upper bound of average network breakdown probability
- Simulation results shows that the upper bound is tight

On going work

- Asymptotic analysis of network breakdown probability under node fault model