Rate-Optimal Streaming Codes over Small Finite Fields for Burst/Random Erasure Channels

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Introduction

Low latency and reliable packet communications are required in various applications.

- Vehicle-to-Vehicle communication
- Real-time applications

Introduction

Low latency and reliable packet communications are required in various applications.

- Vehicle-to-Vehicle communication
- Real-time applications

A solution for this problem is streaming code

Streaming code [Martinian-Sundberg-2004]

Erasure correcting code satisfying decoding delay constraint

Topic of this talk

Construction of streaming codes

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
- Each window contains $\leq a$ random erasures or a burst erasure of length $\leq b$

Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Burst erasure of length 4

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
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Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Burst erasure of length 3

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
- Each window contains $\leq a$ random erasures or a burst erasure of length $\leq b$

Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Burst erasure of length 2

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
- Each window contains $\leq a$ random erasures or a burst erasure of length $\leq b$

Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Random erasure of size 2

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
- Each window contains $\leq a$ random erasures or a burst erasure of length $\leq b$

Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Random erasure of size 1

Channel Model

(a, b, w)-sliding window erasure (SWE) channel

Packet erasure channel with random and burst erasures

- w: width of window
- Each window contains $\leq a$ random erasures or a burst erasure of length $\leq b$

Example of Erasure Pattern: $a = 2$, $b = 4$, $w = 5$

Transmitted Packets

$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$	$\boldsymbol{x}_{10}$

Received Packets

?	?	?	?	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	?	$\boldsymbol{x}_9$	?

Random erasure of size 2

Decoding Delay Constraint

Streaming codes impose the τ-decoding delay constraint on the decoder.
Packet $\boldsymbol{x}_i$ must be recovered from received packets $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{i+\tau}$

$(a, b, w; \tau)$ delay-constrained sliding-window (DCSW) model

- (a, b, w)-SWE channel
- τ-decoding delay (DD) constraint

Optimal rate of streaming code for $(a, b, w; \tau)$ DCSW model [Badr-2017]

$\tau_{\text{eff}} := \min\{w - 1, \tau\}$

$$R \leq \frac{\tau_{\text{eff}} - a + 1}{\tau_{\text{eff}} - a + b + 1}$$

The streaming codes achieving this equality are called rate-optimal.

In this talk,

We construct rate-optimal streaming codes for $(a, b, w; \tau)$ DCSW model.

Related Works

Rate Optimal Streaming Codes for $(a, b, w; \tau)$ DCSW model

Paper	Order of GF	Condition of Parameter	Explicit
[Fong-2019]	$2\binom{\tau+1}{a}$	Arbitrary feasible param.	No
[Krishnan-2020]	$\bar{\tau}^2$	Arbitrary feasible param.	No
[Krishnan-2020]	$\bar{\tau}$	$b - a = 1$	No
[Krishnan-2020]	$\bar{\tau}$	$(\tau + a + 1) \geq 2b \geq 4a$	Yes
[Krishnan-2020]	$\overline{(\frac{a}{b}\tau)}$	$a \mid b \mid (\tau + 1 + b + a)$	Yes
[Ramkumar-2020]	$\bar{\tau}$	$\gcd(b, \tau + 1 - a) \geq a$	Yes
[Ramkumar-2020]	$\bar{\tau}$	$[\tau + 1 \geq a + b] \wedge [a \vert b]$	Yes
[Ramkumar-2020]	$\bar{\tau}$	$[\tau + 1 \geq a + b] \wedge [a \vert (b + 1)]$	Yes
[Domanovitz-2022]	$\bar{\tau}^2$	Arbitrary feasible param.	Yes
[Bhatnagar-2022]	$\bar{\tau}$	$\tau \geq w + b - a - 1$	Yes
This work	$\bar{\tau}$	Arbitrary feasible param.	Yes

($\bar{\tau}$ represents the integer satisfying $\bar{\tau} \geq \tau$ and $\bar{\tau} = p^m$)

Constirubtion of This Work

- The code construction in this work is influenced by [Krishman-2020] and [Domanovitz-2022].
- These constructions locally use the property of $\mathrm{GF}(\bar{\tau}^2)$ (Almost parts use the property of $\mathrm{GF}(\bar{\tau})$)

Contribution of this work

- By designing parity check matrix,
  - we give the explicit construction of the streaming code
  - we give an efficient encoding algorithm for the code
- By using companion matrix, we reduce the order of finite field.
- We give an efficient decoding algorithm for the code.

Settings of Parameters

In the $(a, b, w; \tau)$ DCSW model, it is sufficient to consider only the case

- $a \leq b$
- $w = \tau + 1$

Reasons:

- If $a > b$, the problem is reduced to the case $a = b$
  (a: number of random erasures, b: length of burst erasure)
- Optimal code for $(a, b, \tau + 1; \tau)$ DCSW model is also optimal for $w \neq \tau + 1$ cases

Optimal rate of $(a, b, w; \tau)$-DCSW model

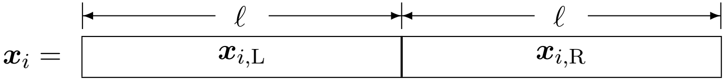
$$\frac{\tau_{\text{eff}} - a + 1}{\tau_{\text{eff}} - a + b + 1} \qquad \tau_{\text{eff}} := \min\{w - 1, \tau\}$$

Sub-packet Division

$\boldsymbol{x}_i \in \mathbb{F}_q^{2\ell}$: ith transmitted packet ($q = 2^m$, $q > \tau$)

Sub-packet Division

We divide each packet into two sub-packets



Parity Check Matrix of Proposed Code

Example: $a = 5, b = 8, \tau = 13, w = 14 \ (\delta := b - a)$

$$
\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{\mathbf{C} \otimes \mathbf{I}} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

$$
\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

where $\beta \in \mathbb{F}_q \setminus \{\gamma + \gamma^{-1} \mid \gamma \in \mathbb{F}_q^*\}$.

$f(x) := x^2 + \beta x + 1$ is an irreducible polynomial over $\mathbb{F}_q$
$\mathbf{J}$ is the companion matrix of $f(x)$

Parity Check Matrix of Proposed Code

Example: $a = 5, b = 8, \tau = 13, w = 14$ $(\delta := b - a)$

$$\mathbf{H} = \left(\begin{array}{ccccccccc|cccccccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{\mathbf{C} \otimes \mathbf{I}} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)$$

Note: Any $t \times t$ sub-matrix of Cauchy matrix $\mathbf{C}$ is non-singular

$$\mathbf{C} \otimes \mathbf{I} = \begin{pmatrix}
c_{1,1}\mathbf{I} & c_{1,2}\mathbf{I} & \cdots & c_{1,\tau+a+1}\mathbf{I} \\
c_{2,1}\mathbf{I} & c_{2,2}\mathbf{I} & \cdots & c_{2,\tau+a+1}\mathbf{I} \\
\vdots & \vdots & \ddots & \vdots \\
c_{a,1}\mathbf{I} & c_{a,2}\mathbf{I} & \cdots & c_{a,\tau+a+1}\mathbf{I}
\end{pmatrix}$$

Parity Check Matrix of Proposed Code

Example: $a = 5, b = 8, \tau = 13, w = 14$ $(\delta := b - a)$

$$\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{\mathbf{C} \otimes \mathbf{I}} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)$$

H satisfies optimal rate

code length $n := \tau - a + b + 1$, dimension $k := n - b = \tau - a + 1$

Optimal rate of DCSW model

$$\frac{\tau_{\text{eff}} - a + 1}{\tau_{\text{eff}} - a + b + 1} \qquad \tau_{\text{eff}} := \min\{w - 1, \tau\}$$

Encoding Algorithm for Proposed Code

Decide parity packets $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_b$ from source packets $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

$$\mathbf{H} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \mathbf{O}_{2n \times \ell}.$$

$$\mathbf{U} = \begin{pmatrix} \boldsymbol{u}_{1,\mathrm{L}} \\ \boldsymbol{u}_{1,\mathrm{R}} \\ \boldsymbol{u}_{2,\mathrm{L}} \\ \boldsymbol{u}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{u}_{k,\mathrm{L}} \\ \boldsymbol{u}_{k,\mathrm{R}} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \boldsymbol{p}_{1,\mathrm{L}} \\ \boldsymbol{p}_{1,\mathrm{R}} \\ \boldsymbol{p}_{2,\mathrm{L}} \\ \boldsymbol{p}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{p}_{b,\mathrm{L}} \\ \boldsymbol{p}_{b,\mathrm{R}} \end{pmatrix}$$

Encoding Algorithm for Proposed Code

Example: $a = 5, b = 8, \tau = 13, w = 14$

$$
\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

Parity part

[Note] There are no effect of sub-packet division to the lower a rows

(1) Generate $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4$ $\quad \boldsymbol{p}_i = \sum_{j=1}^{9} c_{i+1,j} \boldsymbol{u}_j$

(2) Generate $\boldsymbol{p}_5, \boldsymbol{p}_6, \boldsymbol{p}_7$ $\quad \begin{pmatrix} \boldsymbol{p}_{5,\mathrm{L}} \\ \boldsymbol{p}_{5,\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_{1,\mathrm{L}} \\ \boldsymbol{u}_{1,\mathrm{R}} \end{pmatrix} + \begin{pmatrix} \beta & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{9,\mathrm{L}} \\ \boldsymbol{u}_{9,\mathrm{R}} \end{pmatrix}$

(3) Generate $\boldsymbol{p}_8$ $\quad \boldsymbol{p}_8 = \sum_{j=1}^{9} c_{1,j} \boldsymbol{u}_j + \boldsymbol{p}_5$

Examples of Decoding for Proposed Code (Burst 1)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 1–8)

$$
\mathbf{H} = \left(\begin{array}{cccccccccccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{8}{c}{\mathbf{C} \otimes \mathbf{I}} & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

$$
\begin{array}{ccccccccccccccccc}
? & ? & ? & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & - & - & -
\end{array}
$$

Decoding of $\boldsymbol{x}_1$:

- From the property of SWE channel, $\boldsymbol{x}_9, \boldsymbol{x}_{10}, \ldots, \boldsymbol{x}_{14}$ are correctly received (!)
- By the τ-DD constraint, $\boldsymbol{x}_{15}$,$\boldsymbol{x}_{16}$,$\boldsymbol{x}_{17}$ have not been received $(-)$

From constraint of 1st row
$$\begin{pmatrix} \boldsymbol{x}_{1,\mathrm{L}} \\ \boldsymbol{x}_{1,\mathrm{R}} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \boldsymbol{x}_{9,\mathrm{L}} \\ \boldsymbol{x}_{9,\mathrm{R}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}_{14,\mathrm{L}} \\ \boldsymbol{x}_{14,\mathrm{R}} \end{pmatrix}$$

Examples of Decoding for Proposed Code (Burst 1)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 1–8)

$$\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)$$

$$\begin{array}{ccccccccccccccccc} ! & ? & ? & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & ! & - & - \end{array}$$

Decoding of $\boldsymbol{x}_2$:

- From the property of SWE channel, $\boldsymbol{x}_9, \boldsymbol{x}_{10}, \ldots, \boldsymbol{x}_{15}$ are correctly received (!)
- By the τ-DD constraint, $\boldsymbol{x}_{16}$,$\boldsymbol{x}_{17}$ have not been received $(-)$

From constraint of 2nd row
$$\begin{pmatrix} \boldsymbol{x}_{2,\mathrm{L}} \\ \boldsymbol{x}_{2,\mathrm{R}} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \boldsymbol{x}_{10,\mathrm{L}} \\ \boldsymbol{x}_{10,\mathrm{R}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}_{15,\mathrm{L}} \\ \boldsymbol{x}_{15,\mathrm{R}} \end{pmatrix}$$

Examples of Decoding for Proposed Code (Burst 1)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 1–8)

$$
\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

$$
\begin{array}{ccccccccccccccccc}
! & ! & ? & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & ! & ! & -
\end{array}
$$

Decoding of $\boldsymbol{x}_3$:

- From the property of SWE channel, $\boldsymbol{x}_9, \boldsymbol{x}_{10}, \dots, \boldsymbol{x}_{16}$ are correctly received (!)
- By the τ-DD constraint, $\boldsymbol{x}_{17}$ have not been received $(-)$

From constraint of 3rd row
$$
\begin{pmatrix} \boldsymbol{x}_{3,\mathrm{L}} \\ \boldsymbol{x}_{3,\mathrm{R}} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \boldsymbol{x}_{11,\mathrm{L}} \\ \boldsymbol{x}_{11,\mathrm{R}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}_{16,\mathrm{L}} \\ \boldsymbol{x}_{16,\mathrm{R}} \end{pmatrix}
$$

Examples of Decoding for Proposed Code (Burst 1)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 1–8)

$$\mathbf{H} = \left(\begin{array}{ccccccccccccccccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)$$

$$\begin{array}{ccccccccccccccccc}
! & ! & ! & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & ! & ! & !
\end{array}$$

Decoding of $\boldsymbol{x}_4, \boldsymbol{x}_5, \ldots, \boldsymbol{x}_8$

- From the property of SWE channel, $\boldsymbol{x}_9, \boldsymbol{x}_{10}, \ldots, \boldsymbol{x}_{17}$ are correctly received (!)

Since the light blue part is non-singular, these are recoverable

Examples of Decoding for Proposed Code (Burst 2)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 3–10)

$$
\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

$$
\begin{array}{ccccccccccccccccc}
! & ! & ? & ? & ? & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & -
\end{array}
$$

Decoding of $\boldsymbol{x}_3, \boldsymbol{x}_9, \boldsymbol{x}_{10}$:

- From the property of SWE channel, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \ldots, \boldsymbol{x}_{16}$ are correctly received (!)
- By the τ-DD constraint, $\boldsymbol{x}_{17}$ has not been received ($-$)

- By constraint of the 3rd row, recover $\boldsymbol{x}_3$
- By constraint of the 1st row, recover $\boldsymbol{x}_9$
- By constraint of the 2nd row, recover $\boldsymbol{x}_{10}$

Examples of Decoding for Proposed Code (Burst 2)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 3–10)

$$\mathbf{H}=\begin{array}{c}\left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C}\otimes\mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right) \\
\begin{array}{ccccccccccccccccc}
! & ! & ! & ? & ? & ? & ? & ? & ! & ! & ! & ! & ! & ! & ! & ! & !
\end{array}\end{array}$$

Decoding of $\boldsymbol{x}_4, \boldsymbol{x}_5, \ldots, \boldsymbol{x}_8$

- From the property of SWE channel, $\boldsymbol{x}_{17}$ is correctly received (!)

Since the light blue part is non-singular, these are recoverable

Examples of Decoding for Proposed Code (Burst 3)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 9–16)

$$\mathbf{H} = \begin{array}{c}
\left(\begin{array}{ccccccccccccccccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c}{\mathbf{C} \otimes \mathbf{I}} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right) \\
\begin{array}{ccccccccccccccccc}
! & ! & ! & ! & ! & ! & ! & ! & ? & ? & ? & ? & ? & ? & ? & ? & !
\end{array}
\end{array}$$

Examples of Decoding for Proposed Code (Burst 3)

Example: $a = 5, b = 8, \tau = 13, w = 14$ (Burst erasure in 9–16)

$$
\begin{array}{c}
\mathbf{H} = \left(\begin{array}{ccccccccc|ccccc|ccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
\multicolumn{9}{c|}{} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
\multicolumn{9}{c|}{} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{\mathbf{C} \otimes \mathbf{I}} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\multicolumn{9}{c|}{} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right) \\
\begin{array}{ccccccccccccccccc}
! & ! & ! & ! & ! & ! & ! & ! & ? & ? & ? & ? & ? & ? & ? & ? & !
\end{array}
\end{array}
$$

1. Light blue part is non-singular (Shown later)
   $\boldsymbol{x}_9, \boldsymbol{x}_{10}, \boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \boldsymbol{x}_{13}$ are recoverable
2. From 2nd and 3rd rows, $\boldsymbol{x}_{14}, \boldsymbol{x}_{15}$ are recoverable

Principle of Decoding

Theorem 1

$$\left(\begin{array}{c|ccc} \mathbf{J} + a_{1,1}\mathbf{I} & a_{1,2}\mathbf{I} & \cdots & a_{1,t}\mathbf{I} \\ \hline a_{2,1}\mathbf{I} & a_{2,2}\mathbf{I} & \cdots & a_{2,t}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t,1}\mathbf{I} & a_{t,2}\mathbf{I} & \cdots & a_{t,t}\mathbf{I} \end{array}\right)$$

is non-singular if and only if

$$\begin{pmatrix} a_{2,2} & \cdots & a_{2,t} \\ \vdots & \ddots & \vdots \\ a_{t,2} & \cdots & a_{t,t} \end{pmatrix}$$

is non-singular

[Note] Light blue part in the previous slide satisfies the condition

Examples of Decoding for Proposed Code (Random 1)

Example $a = 5, b = 8, \tau = 13, w = 14$

$$
\mathbf{H} = \left(\begin{array}{ccccccccccccccccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right)
$$

$$
\begin{array}{ccccccccccccccccc}
? & ! & ! & ! & ? & ! & ? & ! & ? & ! & ? & ! & ! & ! & - & - & -
\end{array}
$$

From Theorem 1, light blue part is non-singular

Examples of Decoding for Proposed Code (Random 2)

Example: $a = 5, b = 8, \tau = 13, w = 14$

$$\mathbf{H} = \begin{array}{c}
\left(\begin{array}{ccccccccccccccccc}
\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{J} & \mathbf{O} \\
\hline
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\
 & & & & & & & & & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & \mathbf{C} \otimes \mathbf{I} & & & & & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
 & & & & & & & & & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right) \\
\begin{array}{ccccccccccccccccc}
? & ! & ! & ! & ? & ! & ? & ! & ? & ! & ! & ! & ! & ! & - & - & -
\end{array}
\end{array}$$

1. By the lower 4 rows, recover except x_{14}
2. By the 1st row, recover x_{14}

Low-Complexity Decoding Algorithm

Decoding algorithm needs to solve $\mathbf{X}$ of the following equation

Question: Can we efficiently solve the following equation?
(In a naive algorithm, we need to use augmented matrix with $2t$ rows)

$$\begin{pmatrix} \mathbf{J}+a_{1,1}\mathbf{I} & a_{1,2}\mathbf{I} & \cdots & a_{1,t}\mathbf{I} \\ a_{2,1}\mathbf{I} & a_{2,2}\mathbf{I} & \cdots & a_{2,t}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t,1}\mathbf{I} & a_{t,2}\mathbf{I} & \cdots & a_{t,t}\mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{1,\mathrm{L}} \\ \boldsymbol{x}_{1,\mathrm{R}} \\ \boldsymbol{x}_{2,\mathrm{L}} \\ \boldsymbol{x}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{x}_{t,\mathrm{L}} \\ \boldsymbol{x}_{t,\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_{1,\mathrm{L}} \\ \boldsymbol{b}_{1,\mathrm{R}} \\ \boldsymbol{b}_{2,\mathrm{L}} \\ \boldsymbol{b}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{b}_{t,\mathrm{L}} \\ \boldsymbol{b}_{t,\mathrm{R}} \end{pmatrix}$$

Low-Complexity Decoding Algorithm

Decoding algorithm needs to solve $\mathbf{X}$ of the following equation

Question: Can we efficiently solve the following equation?
(In a naive algorithm, we need to use augmented matrix with $2t$ rows)

$$\begin{pmatrix} \mathbf{J} + a_{1,1}\mathbf{I} & a_{1,2}\mathbf{I} & \cdots & a_{1,t}\mathbf{I} \\ a_{2,1}\mathbf{I} & a_{2,2}\mathbf{I} & \cdots & a_{2,t}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t,1}\mathbf{I} & a_{t,2}\mathbf{I} & \cdots & a_{t,t}\mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{1,\mathrm{L}} \\ \boldsymbol{x}_{1,\mathrm{R}} \\ \boldsymbol{x}_{2,\mathrm{L}} \\ \boldsymbol{x}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{x}_{t,\mathrm{L}} \\ \boldsymbol{x}_{t,\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_{1,\mathrm{L}} \\ \boldsymbol{b}_{1,\mathrm{R}} \\ \boldsymbol{b}_{2,\mathrm{L}} \\ \boldsymbol{b}_{2,\mathrm{R}} \\ \vdots \\ \boldsymbol{b}_{t,\mathrm{L}} \\ \boldsymbol{b}_{t,\mathrm{R}} \end{pmatrix}$$

Answer: It can be solved like as the following problem (matrix of t rows)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,t} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t,1} & a_{t,2} & \cdots & a_{t,t} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_t \end{pmatrix}$$

Low-Complexity Decoding Algorithm

1. Construct augmented matrix $\mathbf{F}$

$$\mathbf{F} = \left(\begin{array}{cccc|cc} a_{1,1} & a_{1,2} & \cdots & a_{1,t} & \boldsymbol{b}_{1,\mathrm{L}} & \boldsymbol{b}_{1,\mathrm{R}} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,t} & \boldsymbol{b}_{2,\mathrm{L}} & \boldsymbol{b}_{2,\mathrm{R}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{t,1} & a_{t,2} & \cdots & a_{t,t} & \boldsymbol{b}_{t,\mathrm{L}} & \boldsymbol{b}_{t,\mathrm{R}} \end{array}\right).$$

2. (Forward Elimination) Apply elementary row operations to $\mathbf{F}$

$$\left(\begin{array}{c|ccc|cc} a'_{1,1} & 0 & \cdots & 0 & \boldsymbol{b}'_{1,\mathrm{L}} & \boldsymbol{b}'_{1,\mathrm{R}} \\ \cline{2-4} a'_{2,1} & & & & \boldsymbol{b}'_{2,\mathrm{L}} & \boldsymbol{b}'_{2,\mathrm{R}} \\ \vdots & & \mathbf{I} & & \vdots & \vdots \\ a_{t,1'} & & & & \boldsymbol{b}'_{t,\mathrm{L}} & \boldsymbol{b}'_{t,\mathrm{R}} \end{array}\right).$$

3. Derive $\boldsymbol{x}_{1,\mathrm{L}}, \boldsymbol{x}_{1,\mathrm{R}}$ as follows:

$$\begin{pmatrix} \boldsymbol{x}_{1,\mathrm{L}} \\ \boldsymbol{x}_{1,\mathrm{R}} \end{pmatrix} = (f(a'_{1,1}))^{-1} \begin{pmatrix} \beta + a'_{1,1} & 1 \\ 1 & a'_{1,1} \end{pmatrix} \begin{pmatrix} \boldsymbol{b}'_{1,\mathrm{L}} \\ \boldsymbol{b}'_{1,\mathrm{R}} \end{pmatrix}.$$

4. (Backward Substitution) Derive $\boldsymbol{x}_{i,\mathrm{L}}, \boldsymbol{x}_{i,\mathrm{R}}$ $(i = 2, \ldots, k)$ as follows:

$$(\boldsymbol{x}_{i,\mathrm{L}}, \boldsymbol{x}_{i,\mathrm{R}}) = (\boldsymbol{b}'_{i,\mathrm{L}}, \boldsymbol{b}'_{i,\mathrm{R}}) + a'_{i,1}(\boldsymbol{x}_{1,\mathrm{L}}, \boldsymbol{x}_{1,\mathrm{R}}).$$

Conclusion

Construct streaming codes satisfying

- Rate-optimal
- Over $\mathbb{F}_{\bar{\tau}}$ ($\bar{\tau} > \tau$ and $\bar{\tau} = 2^m$)
- Explicit construction

For this code,

- we give an encoding algorithm
- we give a decoding algorithm
- we reduce the complexity of sub-routine of decoding (Theorem 1)