An Improvement of Non-binary Code Correcting Single b-Burst of Insertions or Deletions

Toyohiko Saeki, Takayuki Nozaki

Yamaguchi University

ISITA2018
Oct. 29th, 2018

Background (1: Purpose)

b-burst insertion/deletion channel

Exactly b consecutive insertions/deletions occurs to $\boldsymbol{x}$

Example: 3-burst deletion

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$$
$$\boldsymbol{y} = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9)$$

Received word $\boldsymbol{y}$ becomes the concatenation of remaining sequence

Purpose of this research

Constructing non-binary (q-ary) code correcting b-burst insertion/deletion

Background (2: Existing works, Contribution, Outline)

Existing works of I/D codes

	Binary	Non-binary
Single I/D	VT code [Varshamov1965] (proved by [Levenshtein1966])	qVT code [Tenengolts1984]
Burst I/D	[Cheng2014] ⇓ Improve [Schoney2017-04]	[Schoney2017-10] ⇓ Improve This work

Contributions of this research

- Constructing non-binary burst I/D code with larger cardinality
- Devising decoding algorithm for the code
- Evaluating number of codewords (Numerical and Asymptotic)

Outline

1. Existing works
2. Code construction
3. Number of Codewords

Existing Work (1: Single I/D codes)

VT code: Binary single I/D code

n: code length, $a \in [n+1] := \{0, 1, 2, \ldots, n\}$

$$\mathrm{VT}_a(n) = \left\{ \boldsymbol{x} \in \{0,1\}^n \mid \sum_{i=1}^{n} i x_i \equiv a \pmod{n+1} \right\}.$$

Non-binary VT code: non-binary single I/D code

q: number of alphabet, $a \in [n], c \in [q]$

$$q\mathrm{VT}_{a,c}(n,q) = \left\{ \boldsymbol{x} \in [q]^n \mid \sum_{i=1}^{n} x_i \equiv c \pmod{q}, \sigma(\boldsymbol{x}) \in \mathrm{VT}_a(n-1) \right\}.$$

Ascent sequence $\sigma(\boldsymbol{x})$ $\sigma : [q]^n \to \{0,1\}^{n-1}$

$$\sigma(\boldsymbol{x})_i = \begin{cases} 1 & (x_i < x_{i+1}) \\ 0 & (x_i \geq x_{i+1}) \end{cases}$$

$$\begin{aligned} \boldsymbol{x} &= 30133212 \\ \sigma(\boldsymbol{x}) &= 0110001 \quad \text{ascent sequence} \end{aligned}$$

Existing Work (2: Burst I/D codes 1)

b-Interleave (Matrix representation of $\boldsymbol{x}$)

$$A_b(\boldsymbol{x}) = \begin{pmatrix} x_1 & x_{b+1} & \cdots & x_{n-b+1} \\ x_2 & x_{b+2} & \cdots & x_{n-b+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_b & x_{2b} & \cdots & x_n \end{pmatrix}.$$

Example: 3-interleaving $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$

$$A_3(\boldsymbol{x}) = \begin{pmatrix} x_1 & x_4 & x_7 & x_{10} \\ x_2 & x_5 & x_8 & x_{11} \\ x_3 & x_6 & x_9 & x_{12} \end{pmatrix}$$

- Each row contains single deletion
- The deletion position of 1st row gives deletion range of other rows

Existing Work (3: Burst I/D codes 2)

Detection of Deletion Position:

Even if we correct a deletion, we cannot detect deletion position

$$\begin{aligned} \boldsymbol{x} &= (1,0,0,1,1,1) \\ \boldsymbol{x}_{\neg 2} &= (1\quad ,0,1,1,1) \\ \boldsymbol{x}_{\neg 3} &= (1,0\quad ,1,1,1) \end{aligned}$$

The received words are same if a deletion occurs in a same run.
One cannot detect which symbol in a run is deleted

run

$$A_b(x) = \begin{pmatrix} \cdots & \cdots & 1 & \boxed{0 \;\; 0 \;\; \cdots \;\; 0} & 1 & \cdots & \cdots \\ \cdots & \cdots & \boxed{* \;\; * \;\; * \;\; \cdots \;\; *} & & * & \cdots & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \cdots & \cdots & \boxed{* \;\; * \;\; * \;\; \cdots \;\; *} & & * & \cdots & \cdots \end{pmatrix}$$

range of deletion position

- We need to limit the run length of 1st row
- We use a code correcting single deletion with given deletion range

Existing Work (4: Burst I/D codes 3)

Framework of b-burst deletion correcting code

$$A_b(\boldsymbol{x}) = \underbrace{\begin{pmatrix} \text{Single I/D code with maximum run length } r \\ (r+1)\text{-bounded single deletion correcting code} \\ \vdots \\ (r+1)\text{-bounded single deletion correcting code} \end{pmatrix}}_{n/b}$$

$(r+1)$-bounded single deletion correcting code

Correcting single deletion with side information of deletion position

(Side information) $(r+1)$-consecutive deletion positions

Existing Work (5: Burst I/D codes 4)

b-burst I/D code [Schonery2017-04]

$$
\begin{aligned}
C_{2,b} = \{\boldsymbol{x} \in \{0,1\}^n \mid & A_b(\boldsymbol{x})_1 \in \mathrm{VT}_a(n/b) \cap S_2(n/b, r), \\
& A_b(\boldsymbol{x})_2 \in \mathrm{SVT}_{d,e}(n/b, r+1), \\
& \vdots \\
& A_b(\boldsymbol{x})_b \in \mathrm{SVT}_{d,e}(n/b, r+1)\}
\end{aligned}
$$

$S_2(n/b, r)$: set of binary sequence of length n/b with maximum run length at most r

Shifted VT (SVT) code

r-bounded single deletion code

$$
\begin{aligned}
\mathrm{SVT}_{d,e}(n,r) = \Big\{\boldsymbol{x} \in \{0,1\}^n \mid & \textstyle\sum_{i=1}^{n} i x_i \equiv d \pmod{r}, \\
& \textstyle\sum_{i=1}^{n} x_i \equiv e \pmod{2}\Big\}
\end{aligned}
$$

$d \in [r], e \in \{0,1\}$

Existing Work (6: Non-binary Burst I/D codes)

Non-binary SVT code

$$q\mathrm{SVT}_{d,e,f}(n,r,q) := \Big\{ \boldsymbol{x} \in [q]^n \mid \textstyle\sum_{i=1}^{n} x_i \equiv f \pmod{q}, \\ \sigma(\boldsymbol{x}) \in \mathrm{SVT}_{d,e}(n-1,r) \Big\}$$

[Schoney2017-11, Lemma 1] Correction capability of NB-SVT code

$q\mathrm{SVT}_{d,e,f}(n,r,q)$ is $(r-1)$-bounded single deletion correcting code

Non-binary b-burst I/D code [Schoney2017-11]

$$\begin{aligned} \check{C}_{q,b} = \{\boldsymbol{x} \in [q]^n \mid & A_b(\boldsymbol{x})_1 \in q\mathrm{VT}_{a,b}(n/b,q) \cap S_q(n/b,r), \\ & A_b(\boldsymbol{x})_2 \in q\mathrm{SVT}_{d,e,f}(n/b, r+2, q), \\ & \vdots \\ & A_b(\boldsymbol{x})_b \in q\mathrm{SVT}_{d,e,f}(n/b, r+2, q)\} \end{aligned}$$

Main Result

Non-binary SVT code

$$q\mathrm{SVT}_{d,e,f}(n,r,q) := \Big\{ \boldsymbol{x} \in [q]^n \mid \textstyle\sum_{i=1}^{n} x_i \equiv f \pmod{q}, \\ \sigma(\boldsymbol{x}) \in \mathrm{SVT}_{d,e}(n-1,r) \Big\}$$

[Theorem 2] Correction capability of NB-SVT code

$q\mathrm{SVT}_{d,e,f}(n,r,q)$ is r-bounded single deletion correcting code

Non-binary b-burst I/D code [This work]

$$C_{q,b} = \{ \boldsymbol{x} \in [q]^n \mid A_b(\boldsymbol{x})_1 \in q\mathrm{VT}_{a,b}(n/b,q) \cap S_q(n/b,r), \\ A_b(\boldsymbol{x})_2 \in q\mathrm{SVT}_{d,e,f}(n/b, r+1, q), \\ \vdots \\ A_b(\boldsymbol{x})_b \in q\mathrm{SVT}_{d,e,f}(n/b, r+1, q) \}$$

(Proof of Theorem 2) Straightforward but long... See arXiv:1804.04824

Number of Codewords (1: Lower bound 1)

$$C_{q,b} = \{\boldsymbol{x} \in [q]^n \mid A_b(\boldsymbol{x})_1 \in q\mathrm{VT}_{a,b}(n/b, q) \cap S_q(n/b, r), \quad A_b(\boldsymbol{x})_i \in q\mathrm{SVT}_{d,e,f}(n/b, r+1, q) \text{ for } i \in [2, b]\}$$

[Lemma 5] Lower bound of # of run length limited sequences

$$|S_{n,q}(r)| \geq (q^r - n)q^{n-r}.$$

[Lemma 6] Lower bound of $|A_b(\boldsymbol{x})_1|$

$$\max_{a\in[n],c\in[q]} |q\mathrm{VT}_{a,c}(n/b, q) \cap S_q(n/b, r)| \geq \frac{(q^r - n)q^{n-r}}{nq} = \frac{(q^r - n)q^{n-r-1}}{n}.$$

[Lemma 7] Lower bound of non-binary SVT code

$$\max_{d\in[r+1],e\in[2],f\in[q]} |q\mathrm{SVT}_{d,e,f}(n, r+1, q)| \geq \frac{q^n}{2(r+1)q} = \frac{q^{n-1}}{2(r+1)}.$$

(Lemma 5) From Union bound

(Lemma 6,7) Pigeonhole principle

Number of Codewords (2: Lower bound 2)

$$C_{q,b} = \{\boldsymbol{x} \in [q]^n \mid A_b(\boldsymbol{x})_1 \in q\mathrm{VT}_{a,b}(n/b, q) \cap S_2(n/b, r), \\ A_b(\boldsymbol{x})_i \in q\mathrm{SVT}_{d,e,f}(n/b, r+1, q) \text{ for } i \in [2, b]\}$$

[Theorem 4] Lower bound of $|C_{q,b}|$

For all r, the cardinality of $C_{q,b}$ satisfies

$$\max |C_{q,b}| \geq \frac{q^{n-b}}{n} \frac{b - nq^{-r}}{2^{b-1}(r+1)^{b-1}}.$$

Substituting $r = \log_q n$, we get

$$\max |C_{q,b}| \geq \frac{2q^{n-b}}{n} \cdot \frac{b-1}{2^b(\log_q n + 1)^{b-1}}.$$

Its redundancy (the number of extra symbols: $n - \log_q |C|$) is

$$b + \log_q n - \log_q(b-1) + (b-1)\log_q 2 + (b-1)\log_q(\log_q n + 1).$$

Number of Codewords (3: Upper bound of the best code)

$M_{q,b}(n)$: q-ary b-burst I/D code with largest cardinality

[Theorem 5]

For enough large n, the following holds:

$$|M_{q,b}(n)| \leq \frac{q^{n-b+1}}{(q-1)n}$$

Upper bound of redundancy for $M_{q,b}(n)$ is

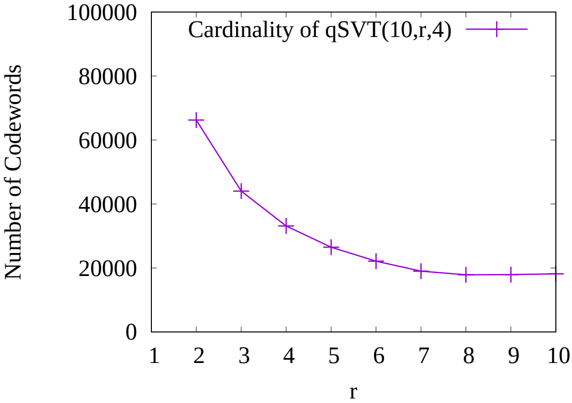
$$b - \log_q 2 + \log_q n$$

Gap of redundancy between $C_{q,b}$ and $M_{q,b}(n)$ is

$$1 - \log_q(q-1) - \log_q(b-1) + (b-1)\log_q 2 + (b-1)\log_q[\log_q n + 1]$$
$$= O(\log_q \log_q n)$$

Fraction of gap of redundancy $O(\frac{\log_q \log_q n}{n})$

Number of Codewords (4: Numerical Example)



$n = 10, q = 4$

The number of codewords decreases for small r

$\Rightarrow$ For $r = \log_q n$, $|q\mathrm{SVT}_{d,e,f}(n, r, q)| > |q\mathrm{SVT}_{d,e,f}(n, r + 1, q)|$

$\Rightarrow$ The proposed code has larger cardinality than [Schoney2017-11]

Conclusion

- We construct non-binary b-burst I/D code
  - We refine the correction capability of non-binary SVT code
- We devise its decoding algorithm
- We evaluate the cardinality of the code